

Chapter 4

Numerical examples

This chapter shows the solution to the same basic cantilever problems using the EFG method; it is just the boundary conditions, which change the problem. It contains two parts of numerical examples. The first part contains two problems whose analytical solution is known. The second one has no information on exact values. The numerical results obtain the good estimations when using two methods above.

4.1 Cantilever beam problem

The solution for cantilever beam subject to end load as shown in Figure 4.1, are given by Timoshenko and Goodier (1970), we also refer to Dolbow and Belytschko (1998), and Liu (2003) as

\[ \sigma_{xx}(x, y) = \frac{P(L-x)y}{I} \]  \hspace{1cm} (4.1)

\[ \sigma_{yy}(x, y) = 0 \]  \hspace{1cm} (4.2)

\[ \tau_{xy}(x, y) = -\frac{P}{2I} \left( \frac{D^2}{4} - y^2 \right) \]  \hspace{1cm} (4.3)

The displacements are given by

\[ u_x = \frac{Py}{6EI} \left[ (6L-3x)x + (2 + \nu)\left( y^2 - \frac{D^2}{4} \right) \right] \]  \hspace{1cm} (4.4)

\[ u_y = -\frac{P}{6EI} \left[ 3\nu y^2 (L-x) + (4 + 5\nu)\frac{D^2 x}{4} + (3L-x)x^2 \right] \]  \hspace{1cm} (4.5)
where moment of inertia, $I$, for a beam with a rectangular cross-section and unit thickness is given by

$$ I = \frac{D^3}{12} \quad (4.6) $$

The displacements (4.4) and (4.5) are prescribed as essential boundary conditions at $x = 0, -\frac{D}{2} \leq y \leq \frac{D}{2}$; the remaining boundaries are traction boundaries.

For convergence studies error norm is defined by

$$ \|E\|_h = \left[ \int_{\Omega} \frac{1}{2} (\varepsilon^h - \varepsilon^{\text{exact}})^T (\sigma^h - \sigma^{\text{exact}}) d\Omega \right]^{\frac{1}{2}} \quad (4.7) $$

The nodal arrangements are shown in Figure 4.2. The following parameters were used for the cantilever beam problem: the dimensions of the beam model are $L = 8$ and $D = 4$. The material properties are: Young’s modulus $E = 3.10^7$, Poisson’s ratio $\nu = 0.3$ and the parabolic shear force $P = 250$. The regular mesh of nodes and the background mesh used to integrate the Galerkin weak form in Chapter three are shown in Figures 4.2 and 4.3 below. In each integration cell, the Gauss quadrature was used to evaluate the stiffness and flexibility matrices of the EFG. The support of the weight function was studied with respect to two models. The rates of convergence were obtained for various values of the scaling parameter $\alpha$. Together there is no body force and plane stress conditions are assumed.
4.1.1 Set of nodes

(a) 17x9

(b) 33x17

(c) 49x25

(d) 65x33

Figure 4.2: Regular nodal arrangements used for the beam problem
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Figure 4.3: Regular meshes were used with 4x4 Gauss points in each cell.

For example, the arrangement of nodes and quadrature cells are shown in Figure 4.3. In each quadrature cell, $4 \times 4$ Gauss points are used. The solutions are obtained to use a quadratic basis function with quartic spline weight function, and support domains of $\alpha = 1.4$ are used.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>$u_y$ Exact ($10^{-2}$)</th>
<th>$u_y$ Meshless ($10^{-2}$)</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>153</td>
<td>-0.3125</td>
<td>-0.3109412014563</td>
<td>0.498816</td>
</tr>
<tr>
<td>561</td>
<td>-0.3125</td>
<td>-0.3120589750140</td>
<td>0.141128</td>
</tr>
<tr>
<td>1225</td>
<td>-0.3125</td>
<td>-0.3124078546524</td>
<td>0.029487</td>
</tr>
<tr>
<td>2145</td>
<td>-0.3125</td>
<td>-0.3124732542642</td>
<td>0.008558</td>
</tr>
</tbody>
</table>

Table 4.1: Comparison of vertical displacement at end of beam (point A)

Table 4.1 compares the EFG calculation for the vertical displacement at the point A on the beam in Figure 4.1 with the exact vertical displacement given in equations (4.4) and (4.5). This calculation was performed for models discretized with 153, 561, 1225 and 2145 nodes.

Figure 4.4: Analytical and meshless numerical solutions for the deflection of the beam in case 153 nodes
The Table 4.1 and Figure 4.4 showed excellent agreements between the exact solution and the EFG solution for the beam deflection along the x-axis. In Figure 4.5, convergence rate of error of vertical displacement of beam at point A is also shown again.

![Error of vertical displacement at the point A](image)

**Figure 4.5:** Convergence rate of error of vertical displacement of cantilever beam at point A

### 4.1.2 The distribution of stress in beam

In Figures 4.6a and 4.7c, they illustrated the comparison between the stresses calculated at the center of the beam and the exact stresses (4.1) and (4.3) by displacement model. Figures 4.6b and 4.7d also illustrated the comparison of stress components by equilibrium model in case 561 nodes.

![Normal stress \( \sigma_{xx} \) by the displacement displacement at mid-beam](image)

![Normal stress \( \sigma_{xx} \) by the equilibrium model at mid-beam](image)

**Figure 4.6:** (a) Normal stress \( \sigma_{xx} \) by the displacement displacement at mid-beam

(b) Normal stress \( \sigma_{xx} \) by the equilibrium model at mid-beam
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(c) Shear stress $\tau_{xy}$ by the displacement model at mid-beam

(d) Shear stress $\tau_{xy}$ by the equilibrium model at mid-beam

These plots show excellent agreement between the EFG results of two models and analytical solution by the stresses distribution at the middle beam. As results show in Figure 4.6, we can see the values of normal stresses of displacement model are better than the equilibrium model on the boundaries with the same of the mesh. And in Figure 4.7, the values of shear stresses on the boundaries are more correctly for equilibrium model. Of particular note is the fact that the EFG solution approximates the natural boundary condition at the free surface of the beam very well. This is a characteristic unreadily achievable with finite elements.

Figure 4.8: Normal and shear stresses at mid-beam with respect to the coarsest set of nodes (45 nodes)

In Figures 4.8, we can see that the displacement model seems better than equilibrium model in the coarsest set of nodes (45 nodes). But when we increase the fine set of nodes (1225 nodes), both models are given good results as Figures 4.9 follows.
4.1.3 Dual analysis

The results for various nodal spacing can be found in Table 4.2 and Figure 4.10. As predicted by the theory, the values of \(-E_T(u_h)\) and \(E_C(\sigma_h)\) converge towards each other; \(E_C(\sigma_h)\) converge with decreasing values and \(-E_T(u_h)\) converge increasing values. It means that the equilibrium model is obtained upper bound and the displacement model is obtained lower bound of exact solution.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Nodal spacing</th>
<th>Exact solution</th>
<th>Displacement model (-E_T)</th>
<th>Equilibrium model (E_C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>17 x 9 = 153</td>
<td>0.5</td>
<td>0.03829166666666</td>
<td>0.038197525993</td>
<td>0.038543315</td>
</tr>
<tr>
<td>33 x 17 = 561</td>
<td>0.25</td>
<td>0.03829166666666</td>
<td>0.038256976729</td>
<td>0.038355617</td>
</tr>
<tr>
<td>49 x 25 = 1225</td>
<td>0.167</td>
<td>0.03829166666666</td>
<td>0.038277575974</td>
<td>0.038310350</td>
</tr>
<tr>
<td>65 x 33 = 2145</td>
<td>0.125</td>
<td>0.03829166666666</td>
<td>0.038290858380</td>
<td>0.038302465</td>
</tr>
</tbody>
</table>

Table 4.2: Results of energy the beam problem
Moreover, these results are in close agreement with the exact result 0.038291666666 obtained by analytical solution. The results of error in energy between two models and analytical solution are presented in Table 4.3 and Figure 4.11.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Nodal spacing</th>
<th>Error in strain energy by displacement model ($10^{-4}$)</th>
<th>Error in complementary strain energy by equilibrium model ($10^{-4}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>17 x 9 = 153</td>
<td>0.5</td>
<td>1.115800</td>
<td>2.51650</td>
</tr>
<tr>
<td>33 x 17 = 561</td>
<td>0.25</td>
<td>0.170610</td>
<td>0.63950</td>
</tr>
<tr>
<td>49 x 25 = 1225</td>
<td>0.167</td>
<td>0.039087</td>
<td>0.18683</td>
</tr>
<tr>
<td>65 x 33 = 2145</td>
<td>0.125</td>
<td>0.008082</td>
<td>0.10798</td>
</tr>
</tbody>
</table>

Table 4.3: Results of error in energy

Figure 4.10: Convergence curves for dual analysis of the cantilever beam

Figure 4.11: Convergence curves of error in energy
4.1.4 Solution with Different Support Sizes

The solutions were obtained for different weight function support sizes (based on the domain of influence multipliers, the scaling parameter $\alpha$ as in Chapter 2). The other parameters were held fixed.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Displacement model $-E_T$</th>
<th>Equilibrium model $E_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$17 \times 9 = 153$</td>
<td>0.03819752599 0.038543315</td>
<td>0.0384833150 0.0384545680</td>
</tr>
<tr>
<td>$33 \times 17 = 561$</td>
<td>0.03825697672 0.038365617</td>
<td>0.0383495617 0.0383396481</td>
</tr>
<tr>
<td>$49 \times 25 = 1225$</td>
<td>0.03828775797 0.038320350</td>
<td>0.0383126350 0.0383106350</td>
</tr>
<tr>
<td>$65 \times 33 = 2145$</td>
<td>0.03829085838 0.038302465</td>
<td>0.0383022465 0.0383022465</td>
</tr>
</tbody>
</table>

Table 4.4: Results of the clamped beam problem with different support sizes

![Convergence curves for the beam problem](image)

Figure 4.12: Convergence curves for dual analysis of the cantilever beam

The solutions with different support sizes are obtained in Table 4.4 and shown in Figure 4.12 for the cantilever beam. In displacement model, only with $\alpha = 1.4$, it gave the result very well while the equilibrium model whose formulation needs more nodes inside the support domain than the displacement one. These results were quite agreement compare with analytical solution and dual theory. In equilibrium model, the values of the scaling parameter $\alpha$ from 2.5 to 4.0 are given good results. The support size cannot over the value $\alpha$ above, while the displacement model is $\alpha$ from 1.4 to 4.0.
Next, we will study the error in energy-norm also with different support sizes; we used the equation (4.7) for this study. The results gave in Table 4.5 and Figure 4.13 for equilibrium model.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Error in energy-norm by the equilibrium model</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 2.5$</td>
<td>$\alpha = 2.8$</td>
<td>$\alpha = 3.0$</td>
</tr>
<tr>
<td>153</td>
<td>0.028338208</td>
<td>0.023473219</td>
<td>0.018636793</td>
</tr>
<tr>
<td>561</td>
<td>0.021507050</td>
<td>0.014082388</td>
<td>0.015077409</td>
</tr>
<tr>
<td>1225</td>
<td>0.015826378</td>
<td>0.009309977</td>
<td>0.011932304</td>
</tr>
<tr>
<td>2145</td>
<td>0.014007388</td>
<td>0.007850916</td>
<td>0.010582480</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Error in energy-norm by the equilibrium model</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 3.2$</td>
<td>$\alpha = 3.5$</td>
<td>$\alpha = 4.0$</td>
</tr>
<tr>
<td>153</td>
<td>0.017409785</td>
<td>0.014074150</td>
<td>0.012267547</td>
</tr>
<tr>
<td>561</td>
<td>0.010568367</td>
<td>0.009727453</td>
<td>0.008025034</td>
</tr>
<tr>
<td>1225</td>
<td>0.007926144</td>
<td>0.008056321</td>
<td>0.006807590</td>
</tr>
<tr>
<td>2145</td>
<td>0.005880628</td>
<td>0.007843917</td>
<td>0.006418300</td>
</tr>
</tbody>
</table>

Table 4.5: Errors in energy-norm corresponds to each $\alpha$ by the equilibrium model

Figure 4.13: Convergence rates in energy-norm with different support sizes by the equilibrium model

As in Figure 4.13, we can see that with the scaling parameter $\alpha = 3.2$, gave the best convergence in equilibrium model. The same of displacement model, the results gave in Table 4.6 and Figure 4.14 and the best convergence is $\alpha = 2.0$. Furthermore, Figure 4.15 also show the best two scaling parameters of both models.
Table 4.6: Errors in energy-norm corresponds to each $\alpha$ by the displacement model

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Error of energy-norm by the displacement model ($\alpha = 1.4$)</th>
<th>$\alpha = 2.0$</th>
<th>$\alpha = 2.5$</th>
<th>$\alpha = 2.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>153</td>
<td>0.026374459</td>
<td>0.01650257</td>
<td>0.018242833</td>
<td>0.021035412</td>
</tr>
<tr>
<td>561</td>
<td>0.017052866</td>
<td>0.01035031</td>
<td>0.010923858</td>
<td>0.012584457</td>
</tr>
<tr>
<td>1225</td>
<td>0.012555719</td>
<td>0.00776072</td>
<td>0.008221283</td>
<td>0.009526389</td>
</tr>
<tr>
<td>2145</td>
<td>0.009974622</td>
<td>0.00603052</td>
<td>0.006918531</td>
<td>0.008433189</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Error of energy-norm by the displacement model ($\alpha = 3.0$)</th>
<th>$\alpha = 3.2$</th>
<th>$\alpha = 3.5$</th>
<th>$\alpha = 4.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>153</td>
<td>0.023755848</td>
<td>0.027010910</td>
<td>0.032517573</td>
<td>0.041850685</td>
</tr>
<tr>
<td>561</td>
<td>0.014358030</td>
<td>0.016452084</td>
<td>0.019774296</td>
<td>0.024607178</td>
</tr>
<tr>
<td>1225</td>
<td>0.010815990</td>
<td>0.012213879</td>
<td>0.014272337</td>
<td>0.017555744</td>
</tr>
<tr>
<td>2145</td>
<td>0.009593024</td>
<td>0.010694043</td>
<td>0.012204103</td>
<td>0.014846552</td>
</tr>
</tbody>
</table>

Figure 4.14: Convergence rates in energy-norm with different support sizes by the displacement model

A too small support domain leads to large errors, which occur because there are not enough nodes to perform interpolation for the field variable. On the contrary, a too large support domain also leads to large errors, which because of the increasing of the error of the numerical integrations. It might increase the numerical errors and computational cost. Therefore, in equilibrium model, $\alpha = 2.5$ to 4.0 and in displacement model $\alpha = 1.4$ to 4.0 should be chosen for these problems.
Figure 4.15: Case $\alpha = 2.0, 3.2$ by displacement and equilibrium models, respectively

Stresses distribution in beam (Case: $33 \times 17 = 561$ nodes)
Figure 4.16a: Normal stress $\sigma_{xx}$ obtained by equilibrium, displacement models and analytical solution
Figure 4.16b: Normal stress $\sigma_{yy}$ obtained by equilibrium, displacement models and analytic solution
Figure 4.16c: Shear stress $\tau_{xy}$ obtained by equilibrium, displacement models and analytic solution.
4.2 Bending and shearing of rectangular beam

The second numerical example is taken from Beckers, Zhong and Maunder (LTAS-Infographie, University of Liege), the master theses of Nguyen-Tien (1999) and Do-Viet (1999). It is a rectangular cantilever beam under bending and parabolic shear, with only rigid body modes constrained as Figure 4.17. The boundary conditions are such that the exact displacement field can be expressed as

\[ u(x, y) = -\frac{Py}{EI} \left[ \frac{1}{2} (L^2 - x^2) + \frac{1}{6} (2 + \nu) \left( y^2 - \frac{D^2}{4} \right) \right] \] (4.8)

\[ v(x, y) = -\frac{P}{EI} \left[ \frac{1}{3} L^3 - \frac{1}{2} L^2 x + \frac{1}{6} x^3 + \frac{1}{24} (4 + 5\nu) D^2 (L - x) + \frac{\nu}{2} xy^2 \right] \] (4.9)

where

\[ I = \frac{TD^3}{12} \] (4.10)

is the second moment of area of the beam section. The exact stress field is

\[ \sigma_{xx}(x, y) = \frac{P}{I} xy \] (4.11)

\[ \sigma_{yy}(x, y) = 0 \] (4.12)

\[ \tau_{xy}(x, y) = \frac{P}{I} \left( \frac{1}{8} D^2 - \frac{1}{2} y^2 \right) \] (4.13)
The parameters of the structure are: Young’s modulus $E = 3.10^7$, Poisson’s ratios $\nu = 0.3$, parabolic traction is applied to the free end $P = 250$, the height of the beam $D = 4$, the length of the beam $L = 8$ and moment $M = 2000$. There is no body force and plane stress conditions are assumed. The nodes of the meshless approximation are regularly spaced.

We analyse similar to the cantilever problem above, the vertical displacements at the point O are given in Table 4.7 and Figure 4.18.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>$u_y$ Exact $(10^{-3})$</th>
<th>$u_y$ Meshless $(10^{-3})$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>153</td>
<td>-0.3125</td>
<td>-0.31095265238918</td>
<td>0.49515</td>
</tr>
<tr>
<td>561</td>
<td>-0.3125</td>
<td>-0.31213426857831</td>
<td>0.11703</td>
</tr>
<tr>
<td>1225</td>
<td>-0.3125</td>
<td>-0.31243003114316</td>
<td>0.02239</td>
</tr>
<tr>
<td>2145</td>
<td>-0.3125</td>
<td>-0.31248351219841</td>
<td>0.00527</td>
</tr>
</tbody>
</table>

**Table 4.7**: Comparison of vertical displacement at end of beam (point O)

![Figure 4.18](image-url)

**Figure 4.18**: Analytical and meshless numerical solutions for the deflection of the beam

These Table 4.7 and Figure 4.18 show excellent agreements between the analytical and the EFG solution. In Figure 4.19, the convergence rate of error of vertical displacement of beam at point O also shows again.
4.2.1 The distribution of stress in beam

Figures 4.20a and 4.21c illustrated the comparison between the stresses calculated at the center of the beam and the exact stresses (4.11), (4.12) and (4.13) by displacement model. Figures 4.20b and 4.21d also illustrated the comparison of stress components by equilibrium model.
Figure 4.21: (c) Shear stress $\tau_{xy}$ by the displacement model at mid-beam  
(d) Shear stress $\tau_{xy}$ by the stress model at mid-beam

These plots show excellent agreement between the EFG results of two models and analytical solution by the stresses distribution at the middle beam. \textit{Figures 4.22 are shown} the shear and normal stress components at middle beam. Stresses obtain from equilibrium approach are better than the displacement approach.

\textbf{Figure 4.22:} Normal and shear stresses at mid-beam with respect to the fine set of nodes


### Dual analysis

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Nodal spacing</th>
<th>Exact solution</th>
<th>Displacement model $E_U$</th>
<th>Equilibrium model $E_V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17 x 9 = 153</td>
<td>0.5</td>
<td>0.03983333</td>
<td>0.03953726</td>
<td>0.04101767</td>
</tr>
<tr>
<td>33 x 17 = 561</td>
<td>0.25</td>
<td>0.03983333</td>
<td>0.03975440</td>
<td>0.03992104</td>
</tr>
<tr>
<td>49 x 25 = 1225</td>
<td>0.167</td>
<td>0.03983333</td>
<td>0.03981447</td>
<td>0.03993645</td>
</tr>
<tr>
<td>65 x 33 = 2145</td>
<td>0.125</td>
<td>0.03983333</td>
<td>0.03983245</td>
<td>0.03992011</td>
</tr>
</tbody>
</table>

**Table 4.8:** Results of energy the beam problem

Theses results are in close agreement with the exact result 0.03983333 obtained by analytical solution. The results for various nodal spacing can be found in Table 4.8 and Figure 4.23. The values of $E_U(u_h)$ and $E_V(\sigma_h)$ converge towards each other; $E_V(\sigma_h)$ converge with decreasing values and $E_U(u_h)$ converge increasing values. That meant the equilibrium model is obtained upper bound and the displacement model is obtained lower bound of exact solution. We also can see the displacement models converge to the exact solution from below, while equilibrium models converge to the exact solution from above. Moreover, the results of error in energy between two models and analytical solution are presented in Table 4.9 and Figure 4.24.
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<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Nodal spacing</th>
<th>Error in strain energy by displacement model ($10^{-4}$)</th>
<th>Error in complementary strain energy by equilibrium model ($10^{-4}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>17 x 9 = 153</td>
<td>0.5</td>
<td>2.9606800</td>
<td>12.09114</td>
</tr>
<tr>
<td>33 x 17 = 561</td>
<td>0.25</td>
<td>0.7892999</td>
<td>2.620200</td>
</tr>
<tr>
<td>49 x 25 = 1225</td>
<td>0.167</td>
<td>0.1886000</td>
<td>1.034810</td>
</tr>
<tr>
<td>65 x 33 = 2145</td>
<td>0.125</td>
<td>0.0088000</td>
<td>0.867800</td>
</tr>
</tbody>
</table>

Table 4.9: Results of error in energy

![Convergence curves of error in energy](image)

**Figure 4.24:** Convergence curves of error in energy

### 4.2.3 Solution with Different Support Sizes

The solutions with different support sizes are obtained in *Table 4.10* and shown in *Figure 4.25* for the beam problem. The same of the example above, these results were quite agreement compare with analytical solution and dual theory.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Displacement model $E_U$</th>
<th>Equilibrium model $E_V$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 1.4$</td>
<td>$\alpha = 2.5$</td>
</tr>
<tr>
<td>17 x 9 = 153</td>
<td>0.039465525</td>
<td>0.041042444</td>
</tr>
<tr>
<td>33 x 17 = 561</td>
<td>0.039729185</td>
<td>0.040195350</td>
</tr>
<tr>
<td>49 x 25 = 1225</td>
<td>0.03978988</td>
<td>0.039948811</td>
</tr>
<tr>
<td>65 x 33 = 2145</td>
<td>0.039830668</td>
<td>0.039928110</td>
</tr>
</tbody>
</table>

Table 4.10: Results of the clamped beam problem with different support sizes
We will similarly study the error in energy-norm also with different support sizes; we also used the equation (4.7) for this study. The results gave in Table 4.11 and Figure 4.26 for equilibrium model below.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Error in energy-norm by equilibrium model</th>
<th>( \alpha = 2.5 )</th>
<th>( \alpha = 2.8 )</th>
<th>( \alpha = 3.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>153</td>
<td></td>
<td>0.028338208</td>
<td>0.023473291</td>
<td>0.018636793</td>
</tr>
<tr>
<td>561</td>
<td></td>
<td>0.021507050</td>
<td>0.014082388</td>
<td>0.015077409</td>
</tr>
<tr>
<td>1225</td>
<td></td>
<td>0.015826378</td>
<td>0.009309977</td>
<td>0.011932304</td>
</tr>
<tr>
<td>2145</td>
<td></td>
<td>0.014007389</td>
<td>0.007850916</td>
<td>0.010582480</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Error in energy-norm by equilibrium model</th>
<th>( \alpha = 3.2 )</th>
<th>( \alpha = 3.5 )</th>
<th>( \alpha = 4.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>153</td>
<td></td>
<td>0.017409785</td>
<td>0.014074150</td>
<td>0.012267547</td>
</tr>
<tr>
<td>561</td>
<td></td>
<td>0.010568367</td>
<td>0.009727453</td>
<td>0.008025034</td>
</tr>
<tr>
<td>1225</td>
<td></td>
<td>0.007926144</td>
<td>0.008056321</td>
<td>0.006807590</td>
</tr>
<tr>
<td>2145</td>
<td></td>
<td>0.005880628</td>
<td>0.007843917</td>
<td>0.006418300</td>
</tr>
</tbody>
</table>

Table 4.11: Errors in energy-norm corresponds to each \( \alpha \) by equilibrium model
In displacement model, the results are similar to equilibrium model and Table 4.12 and Figure 4.27 give these

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>$\alpha = 1.4$</th>
<th>$\alpha = 2.0$</th>
<th>$\alpha = 2.5$</th>
<th>$\alpha = 2.8$</th>
<th>$\alpha = 3.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>153</td>
<td>0.026397131</td>
<td>0.016685512</td>
<td>0.017654585</td>
<td>0.019805387</td>
<td>0.031232979</td>
</tr>
<tr>
<td>561</td>
<td>0.017069062</td>
<td>0.010455906</td>
<td>0.011807611</td>
<td>0.014050253</td>
<td>0.021333316</td>
</tr>
<tr>
<td>1225</td>
<td>0.012591866</td>
<td>0.008045776</td>
<td>0.009675482</td>
<td>0.011625716</td>
<td>0.016083170</td>
</tr>
<tr>
<td>2145</td>
<td>0.010106229</td>
<td>0.006883395</td>
<td>0.008785686</td>
<td>0.010401235</td>
<td>0.014385402</td>
</tr>
</tbody>
</table>

Table 4.12: Errors in energy-norm corresponds to each $\alpha$ by displacement model
4.2.4 Conclusions

The convergence of the equilibrium and displacement models is studied by using the benchmark beam problems. The study is conducted under exactly the same conditions. Regularly distributed 153 (17 x 9), 561 (33 x 17), 1225 (49 x 25) and 2145 (65 x 33) nodes are used. The convergences with nodes refinement are shown in Figures 4.13, 4.14, 4.26 and 4.27. The energy norm defined by equation (4.7) is computed for different nodal spacing $h$.

The convergence process of the equilibrium model as in Figure 4.13, 4.26 are not as smooth as the displacement model. Not only the parameters $\alpha$ is chosen in the support domain affected, but also the type of weight functions affect the convergence rate and the accuracy. In this thesis, we used the type of weight function that is the circle domain.

Convergence studies on two problems of elastostatics for which closed form solutions are available show that the method exhibits good accuracy. The rates of convergence in the energy norm were in the range of 1.4 to 4.0 of displacement model and from 2.5 to 4.0 of equilibrium model for different sizes of support.

The rates of convergence were compared to those obtained for essential boundary conditions imposed via Lagrange multipliers. These comparison show that the accuracy and agreement of two models.

Furthermore, for equilibrium model, the most of the values of $\alpha$ took from 2.5 to 4.0 are given good results. If we take this interval outside, the results of problem will not good. This model seems to need more nodes inside the support domain than the displacement model. This is a drawback of the equilibrium model. Because the number of nodes inside the support domain is larger, so it will take a lot of the time for calculation.

Nevertheless, our aim was not to replace the classical displacement model with the equilibrium model but rather to obtain a global error estimator based upon the combination of both models.
4.3 Clamped beam with the parabolic force

The next example, we solve the problem of a clamped beam submitted to a parabolic shear force, the show as figure 4.28 below. This example takes from the PhD thesis of Duflot (2004), and Duflot and Nguyen-Dang (2002), (2004). The numerical values are:

- Young’s modulus: $E = 3.10^7$
- Poisson’s ratios: $\nu = 0.3$
- Total force: $P = 250$
- Height of the beam: $D = 4$
- Length of the beam: $L = 8$

There is no body force and plane stress conditions are assumed. The nodes of the meshless approximation are regularly spaced.

![Figure 4.28: The clamped beam submitted to a parabolic force](image)

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Nodal spacing</th>
<th>Zhong’s solution</th>
<th>Displacement model $-E_T$</th>
<th>Equilibrium model $E_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17 x 9 = 153</td>
<td>0.5</td>
<td>0.0393955</td>
<td>0.0392741172</td>
<td>0.039991951</td>
</tr>
<tr>
<td>33 x 17 = 561</td>
<td>0.25</td>
<td>0.0393955</td>
<td>0.0393751196</td>
<td>0.039559487</td>
</tr>
<tr>
<td>49 x 25 = 1225</td>
<td>0.167</td>
<td>0.0393955</td>
<td>0.0393902654</td>
<td>0.039464977</td>
</tr>
<tr>
<td>65 x 33 = 2145</td>
<td>0.125</td>
<td>0.0393955</td>
<td>0.0393948145</td>
<td>0.039424923</td>
</tr>
</tbody>
</table>

Table 4.13: Results of the clamped beam problem with parabolic force

The results show as in Figure 4.29 and Table 4.13, as examples above, are quite agreement with the result 0.0393955 of Zhong based on a finite element method. The results of error in energy are also shown in Figure 4.30 and Table 4.14. The similar of two examples above, the values of $E_U(u_h)$ and $E_V(\sigma_h)$ converge towards each other; $E_V(\sigma_h)$ converges with decreasing values and $E_U(u_h)$ converge increasing values.


Figure 4.29: Convergence curves for the clamped beam with the parabolic force

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Nodal spacing</th>
<th>Error in strain energy by displacement model ($10^{-4}$)</th>
<th>Error in complementary strain energy by equilibrium model ($10^{-4}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>17 x 9 = 153</td>
<td>0.5</td>
<td>1.2138280</td>
<td>5.96450990</td>
</tr>
<tr>
<td>33 x 17 = 561</td>
<td>0.25</td>
<td>0.2038039</td>
<td>1.63986999</td>
</tr>
<tr>
<td>49 x 25 = 1225</td>
<td>0.167</td>
<td>0.0523460</td>
<td>0.69476999</td>
</tr>
<tr>
<td>65 x 33 = 2145</td>
<td>0.125</td>
<td>0.0068550</td>
<td>0.02942301</td>
</tr>
</tbody>
</table>

Table 4.14: Results of error in energy

Figure 4.30: Convergence curves of error in energy
Solution with Different Support Sizes

The solution with different support sizes are obtained in Table 4.15, 4.16 and showed in Figure 4.31 for this problem. These results were quite agreement compare with analytical solution and theory.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Nodal spacing</th>
<th>Total strain energies $E_T$</th>
<th>Total complementary strain energies $E_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\alpha = 2.0$</td>
<td>$\alpha = 2.8$</td>
</tr>
<tr>
<td>153</td>
<td>0.5</td>
<td>0.0391380570</td>
<td>0.039274117</td>
</tr>
<tr>
<td>561</td>
<td>0.25</td>
<td>0.0393024626</td>
<td>0.039375119</td>
</tr>
<tr>
<td>1225</td>
<td>0.167</td>
<td>0.0393727301</td>
<td>0.039390265</td>
</tr>
<tr>
<td>2145</td>
<td>0.125</td>
<td>0.0393939846</td>
<td>0.039394801</td>
</tr>
</tbody>
</table>

Table 4.15: Results of the clamped beam problem depends on each $\alpha$ by displacement model

Table 4.16: Results of the clamped beam problem depends on each $\alpha$ by equilibrium model

Figure 4.31: Convergence curves for the clamped beam depends on each $\alpha$
These plots show excellent agreement between the EFG results of two models by the stresses distribution at the middle beam. In Figures 4.32 and 4.33 are shown the shear and normal stress components at middle beam.

**Figure 4.32**: Normal stresses at middle beam with respect to the coarsest set of nodes (561 nodes)

**Figure 4.33**: Shear stresses at middle beam with respect to the coarsest set of nodes (561 nodes)
4.4 Clamped beam with the uniform force

The final example is the cantilever beam which clamped on one edge, the opposed edge being loaded with a constant shearing $P = 93.75$, as shown in Figure 4.34. There is no body force and plane stress conditions are considered. The numerical values are $L = 8$, $D = 4$ and the parameters of material: Young’s modulus is $E = 3.10^7$, Poisson’s ratio is $\nu = 0.3$. The nodes of the meshless approximation are regular spaced.

![Figure 4.34: The clamped beam submitted to a uniform force](image)

In this example, the exact solution is not available. The estimated exact energy is $0.088851269$ taken from the master theses of Do-Viet (1999) and Nguyen-Tien (1999). The results for various nodal spacing and energies can be found in Table 4.17 and Figure 4.35 follow. The similar of two examples above, the values of $E_U(u_h)$ and $E_V(\sigma_h)$ converge towards each other; $E_V(\sigma_h)$ converges with decreasing values and $E_U(u_h)$ converge increasing values. Moreover, Figures 4.36 and Table 4.18 are shown the convergence rate of error in energy.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Nodal spacing</th>
<th>Estimated exact solution</th>
<th>Displacement model $E_U$</th>
<th>Equilibrium model $E_V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$17 \times 9 = 153$</td>
<td>0.5</td>
<td>0.088851269</td>
<td>0.0882002880145</td>
<td>0.0929291306188</td>
</tr>
<tr>
<td>$33 \times 17 = 561$</td>
<td>0.25</td>
<td>0.088851269</td>
<td>0.0886325633651</td>
<td>0.0899699369217</td>
</tr>
<tr>
<td>$49 \times 25 = 1225$</td>
<td>0.167</td>
<td>0.088851269</td>
<td>0.0888390651454</td>
<td>0.0892565058912</td>
</tr>
<tr>
<td>$65 \times 33 = 2145$</td>
<td>0.125</td>
<td>0.088851269</td>
<td>0.0888456290983</td>
<td>0.0889449829980</td>
</tr>
</tbody>
</table>

*Table 4.17: Results of the clamped beam problem with uniform force*
Figure 4.35: Convergence curves for the clamped beam with the uniform force

Table 4.18: Results of error in energy

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Nodal spacing</th>
<th>Error in strain energy by displacement model (10^{-3})</th>
<th>Error in complementary strain energy by equilibrium model (10^{-3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>17 x 9 = 153</td>
<td>0.5</td>
<td>0.6509809098</td>
<td>4.0778616188</td>
</tr>
<tr>
<td>33 x 17 = 561</td>
<td>0.25</td>
<td>0.2187056349</td>
<td>1.1186679217</td>
</tr>
<tr>
<td>49 x 25 = 1225</td>
<td>0.167</td>
<td>0.0122038546</td>
<td>0.4052368912</td>
</tr>
<tr>
<td>65 x 33 = 2145</td>
<td>0.125</td>
<td>0.0056399017</td>
<td>0.0937139980</td>
</tr>
</tbody>
</table>

Figure 4.36: Convergence curves of error in energy
CHAPTER 4: NUMERICAL EXAMPLES

Solution with Different Support Sizes

The solution with different support sizes are obtained in Table 4.19, 4.20 and showed in Figure 4.37 for this problem. The same of the example above, these results were quite agreement compare with analytical solution and theory.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Strain energies</th>
<th></th>
<th>Complementary strain energies</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 1.4$</td>
<td>$\alpha = 2.0$</td>
<td>$\alpha = 2.8$</td>
<td></td>
</tr>
<tr>
<td>153</td>
<td>0.088200288</td>
<td>0.088220006</td>
<td>0.088504520</td>
<td></td>
</tr>
<tr>
<td>561</td>
<td>0.088632563</td>
<td>0.088610940</td>
<td>0.088754777</td>
<td></td>
</tr>
<tr>
<td>1225</td>
<td>0.088839065</td>
<td>0.088806375</td>
<td>0.088840390</td>
<td></td>
</tr>
<tr>
<td>2145</td>
<td>0.088845629</td>
<td>0.088849382</td>
<td>0.088847394</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.19: Results of the clamped beam problem depends on each $\alpha$ by the displacement model

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>$\alpha = 2.5$</th>
<th>$\alpha = 2.8$</th>
<th>$\alpha = 3.2$</th>
<th>$\alpha = 3.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>153</td>
<td>0.092929131</td>
<td>0.0919024565</td>
<td>0.091589030</td>
<td>0.091068271</td>
</tr>
<tr>
<td>561</td>
<td>0.089969937</td>
<td>0.0896845090</td>
<td>0.089608060</td>
<td>0.089487926</td>
</tr>
<tr>
<td>1225</td>
<td>0.089256500</td>
<td>0.0891669294</td>
<td>0.089165915</td>
<td>0.089201713</td>
</tr>
<tr>
<td>2145</td>
<td>0.088994983</td>
<td>0.0889749100</td>
<td>0.088974910</td>
<td>0.089049171</td>
</tr>
</tbody>
</table>

Table 4.20: Results of the clamped beam problem depends on each $\alpha$ by the equilibrium model

![Figure 4.37](image)

Figure 4.37: Convergence curves for the clamped beam depends on each $\alpha$
In this problem, it is evident from the results shown in Figure 4.35 and Figure 4.37 for the solution with different support sizes. The fact that we are trying to impose a non-zero shear on the right edge of the beam and a zero shear on the top or bottom edges leads to oscillations in the numerical solutions. Indeed we consider the upper right corner. On the right side of the beam, we impose a shear in the y-direction but on the upper side of the beam we do not enforce a shear in the x-direction. So, the tangential forces are not in-equilibrium at this corner.

**Figure 4.38**: The distribution of normal stress $\sigma_{xx}$ in beam obtained by equilibrium model (561 nodes).

**Figure 4.39**: The distribution of shear stress $\tau_{xy}$ in beam obtained by equilibrium model (561 nodes).
The whole error is concentrated on those two right end points. The results of energy are nice as in Figures 4.35 and 4.37. Maybe this result comes from the richness of the approximation. We used the approximation of $C^2$ continuous and holds $x^2$ and $y^2$ monomials. But the possible of this problem is ill posed. At the corners of the right end of the beam, we indeed impose the same component of stress to have two different values. It means that we enforce a non-zero stress field on the vertical right end, and a zero stress field on the horizontal right end.

*Figures 4.38 and 4.39* shows the distributions of stress in beam by equilibrium model. The shear stress is not smooth, the reason of this smooth probably discussed above. Together, *Figures 4.40 and 4.41* also illustrates the distribution of the normal and shear stresses on the cross section at $x = \frac{L}{2}$ of the beam. Both the displacement and equilibrium models solution are plotted together for comparison. Very good agreement is observed between the stresses calculated by the two models.

![Normal stress at mid-beam](image)

*Figure 4.40*: Normal stresses at middle beam with respect to the coarsest set of nodes (561 nodes)
Comparison with results in Finite element method

To compare by the FEM, I need to mesh depending on the results in FEM above. The results are given the Table 4.21 below. So we only consider the convergence rates of the strain energy between methods. The numerical results of strain energy by FEM and Degree 1 – CV1 methods are taken from Do-Viet (1999). Table 4.22 includes the results of strain energy of four methods.

<table>
<thead>
<tr>
<th>Degree of freedom</th>
<th>Exact strain energy</th>
<th>Strain energy of Meshless method</th>
<th>Strain energy of FEM</th>
<th>Strain energy of Degree 1 -CV 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>0.088851269</td>
<td>0.08508795493101</td>
<td>0.072961421</td>
<td>0.053342066</td>
</tr>
<tr>
<td>256</td>
<td>0.088851269</td>
<td>0.08745927365589</td>
<td>0.083847030</td>
<td>0.076015639</td>
</tr>
<tr>
<td>1024</td>
<td>0.088851269</td>
<td>0.08860361683726</td>
<td>0.087416680</td>
<td>0.084976617</td>
</tr>
<tr>
<td>4096</td>
<td>0.088851269</td>
<td>0.08873256254874</td>
<td>0.088448089</td>
<td>0.087738970</td>
</tr>
</tbody>
</table>

Table 4.21: Comparison of the strain energies

<table>
<thead>
<tr>
<th>Degree of freedom</th>
<th>Error of strain energy of Meshless method</th>
<th>Error of strain energy of FEM</th>
<th>Error of strain energy of Degree 1 -CV 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>0.00376331406899</td>
<td>0.01588984</td>
<td>0.035509203</td>
</tr>
<tr>
<td>256</td>
<td>0.00139199534411</td>
<td>0.00500432</td>
<td>0.012835630</td>
</tr>
<tr>
<td>1024</td>
<td>0.0002476521683726</td>
<td>0.00143459</td>
<td>0.003874652</td>
</tr>
<tr>
<td>4096</td>
<td>0.00011870645125</td>
<td>0.00040318</td>
<td>0.001112299</td>
</tr>
</tbody>
</table>

Table 4.22: Errors of strain energy
CHAPTER 4: NUMERICAL EXAMPLES

4.42 Figure: Convergent of the strain energies of Meshless, FEM, and Degree 1 – CV1 methods

This is a small comparable example between meshless, FEM and Degree 1 – CV methods. The results of the meshless method seem to be better than with the FEM and Degree 1 – CV methods. We can see in Figure 4.42 and Tables 4.21 and 4.22. The demonstration is not considered in this thesis.

4.5 Discussions

The numerical results of the applications of the EFG method for dual analysis based on two models are presented in this chapter. Demonstration through four examples above, the aim of my thesis is quite solved clearly. The meshless equilibrium model is emphasized in this thesis that based on a stress approach. We derive the stresses from an Airy stress function. The stresses of two models are compared with respect to analytical solutions. This function is approximated by a linear combination of MLS shape functions.

The convergence rate also studied for both models in sections 4.1 and 4.2. The solutions with different support domain are considered for a global error estimation between classical displacement and equilibrium models. The drawback of the equilibrium model when comparing with the traditional displacement models requires the second-order derivatives of the MLS shape functions.

We may be estimated a global error from the squared distance between the two approximations which refers to dual analysis. The dual analysis consists of treating the same by two ways: the first, it is using a statically admissible model and the second, a kinematically admissible one. Together, the solutions with different support sizes are also considered.