FALL 2008 Analysis Qualifying Examination

PART A: Do all four problems. Justify your steps briefly.

1. Suppose $F_n : [a,b] \to R,\ n = 1,2,3,\cdots$ is a sequence of absolutely continuous functions. Let $g : [a,b] \to R$ be integrable and $\left|\frac{d}{dt}F_n\right| \leq g,\ n = 1,2,3,\cdots$ Suppose $\frac{d}{dt}F_n \to f\ a.e.$ and $F_n(a) \to A$. Prove that $\{F_n\}$ converges to $F\ \forall x \in [a,b]$, and F is absolutely continuous on $[a,b]$.
2. Let $(X,\mathcal{F})$ be a measure space. Let μ, ν be finite measures on $\mathcal{F}$ such that $\nu \ll \mu$. Let $\overline{\mu} = \mu + \nu$. If $\nu(E) = \int_E f d\overline{\mu}$, prove that $0 \leq f < 1\ a.e.$ with respect to the measure μ.
3. Let f be Lebesgue integrable on the real line and let $f\ \geq 0, f\ \in L^1((0,\infty))$. Prove that the function
$$g(t) = \int_{(0,\infty)} f(x)\exp(-tx)dx,\ 0 < t < \infty$$
is differentiable and give dg/dt. Is g differentiable infinitely often on $(0,\infty)$?
4. If (X,μ) is a finite measure space show that
   (*i*) $\rho(f,g) = \int_X \frac{|f-g|}{1+|f-g|}d\mu$ is a metric on the space of complex-valued measurable functions on X.
   (*ii*) Convergence in the metric ρ is the same as convergence in measure, i.e., show that
   $\int_X \frac{|f_n|}{1+|f_n|}d\mu \to 0\ iff\ f_n \to 0$ in measure.

PART B: Do three problems. Justify your steps briefly.

5. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X such that $X_1 = (X, \|\cdot\|_1)$ and $X_2 = (X, \|\cdot\|_2)$ are complete. Suppose $(\|x_n\|_1 \to 0)$ always implies $(\|x_n\|_2 \to 0)$. Show that
   (*i*) convergence in X_1 implies convergence in X_2 and conversely.
   (*ii*) there exist $a > 0$ and $b > 0$ such that $\forall x \in X,\ a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$.
6. Let $S = I + T^*T : H \to H$, where T is a linear and bounded on the complex Hilbert space H. Show that $S^{-1} : S(H) \to H$ exists. Is $S(H) = H$?
7. Let X be a normed space and let $\{x_n\} \subset X$. Suppose $\{x_n\}$ converges to y. Verify that $y \in \overline{span\{x_1,x_2,x_3,\cdots\}}$.
8. Let X and Y be Banach spaces and $T : X \to Y$ an injective bounded linear operator. Show that $T^{-1} : R(T) \to X$ is bounded if and only if $R(T)$ is closed in Y.

North Carolina State University Department of Mathematics

MA 532-732 (Fall 2008)

Ordinary Differential Equations - PhD Qualifying Exam
8:00-11:00 am

There are three parts to this exam - Part I has two problems and Part II & III have three problems each. WORK EXACTLY SEVEN (7) of the EIGHT (8) problems. Indicate clearly which problem you are omitting.

Please write on only one side of your paper and begin each problem on a new page.

I Suppose $A(t)$, $t \geq 0$, is a continuous $N \times N$ matrix valued function on $[0, \infty)$ and consider the homogeneous linear system

$$y_z'(t) = A(t)y_z(t), \quad t \geq 0, \quad y_z(0) = z \in R^N. \tag{1}$$

Let $Y(t)$, $t \geq 0$, be the matrix solution to (1) such that $Y(0) = I$ (I is the $N \times N$ identity matrix). Recall that for each $z \in R^N$, the (unique) solution y_z to (1) is given by $y_z(t) = Y(t)z$, $\forall t \geq 0$.

(1) Show that $Y(t)$ is invertible for all $t \geq 0$.

(2) Suppose that $b : [0, \infty) \to R^N$ is continuous and consider the nonhomogeneous problem

$$u_z'(t) = A(t)u_z(t) + b(t), \quad t \geq 0, \quad u_z(0) = z. \tag{2}$$

Use variation of constants to show that if

(i) $\|Y(t)Y(s)^{-1}\| \leq Ke^{-\alpha(t-s)}$, $\forall t \geq s \geq 0$ where $K \geq 1$ and $\alpha > 0$; and

(ii) $\lim_{t\to+\infty} \|b(t)\| = 0$;

then $\lim_{t\to\infty} \|u_z(t)\| = 0$, $\forall z \in R^N$.

II (1) Consider the system

$$\begin{aligned} x' &= -x - x^2 + xy, \quad x(0) = x_0, \\ y' &= 3y - y^2 - xy, \quad y(0) = y_0, \end{aligned}$$

$t > 0$. Using a separate graph for each critical point, sketch a phase plot locally about the two critical points

(a) $(0, 3)$

(b) $(1, 2)$

Indicate the direction of flow (as t increases) and the slope of appropriate tangent lines.

(2) Consider the system

$$\begin{aligned} x' &= -y^3, \quad x(0) = x_0, \\ y' &= x - y^3, \quad y(0) = y_0, \end{aligned}$$

$t > 0$.

(a) Show that the linearization about the critical point $(0, 0)$ is not stable.

(b) Use a Lyapunov function of the form $V[x, y] = x^m + by^n$ to show that $(0, 0)$ is a stable critical point for the nonlinear system.

(c) Is $(0, 0)$ asymptotically stable for the nonlinear system? Justify your assertion.

(3) Consider the system

$$\begin{aligned} x' &= y + x(1 - x^2 - 2y^2), \quad x(0) = x_0, \\ y' &= -x + y(1 - x^2 - 2y^2), \quad y(0) = y_0, \end{aligned}$$

$t > 0$. Use the function $V[x, y] = x^2 + y^2$ and the Poincaré-Bendixson Theorem to show that every <u>nontrivial</u> solution $x(t)$, $y(t)$ is either periodic or spirals to a periodic solution as $t \to \infty$.

III Suppose $f : R^N \to R^N$ is C^1 and that the solutions u_z to

$$u_z^{'} = f(u_z), \quad u_z(0) = z, \quad t \geq 0 \tag{3}$$

exist (and are unique since f is C^1) on $[0, \infty)$ for all $z \in R^N$. Let $S(t)z = u_z(t)$, $\forall t \geq 0$, $z \in R^N$ and recall the semigroup S is jointly continuous in t and z and satisfies $S(0)z = z$ and $S(t+s)z = S(t)S(s)z$, $\forall t, s \geq 0$ and $z \in R^N$.

(1) If the solution $u_z(t) = S(t)z$ is bounded on $[0, \infty)$, show that the omega limit set $\omega(z)$ is nonempty, closed and (plus) invariant.

(2) Suppose $V : R^N \to [0, \infty)$ is C^1 (hence locally Lipschitz) and

(V1) V is positive definite

(V2) $V_{(x)}^{'}[x] \leq 0$, $\forall x \in R^N$ (recall $V_{(x)}^{'}[x] = \nabla V(x) \bullet f(x)$ in the C^1 case)

(V3) $V[x] \to \infty$ as $\|x\| \to \infty$.

Show that

(a) each solution to (3) is bounded on $[0, \infty)$, and

(b) $\omega(z) \subset \{x \in R^N : V_{(x)}^{'}[x] = 0\}$, $\forall z \in R^N$.

(3) Suppose $m_+[x, f(x)] \leq -\alpha\|x\| + \beta$, $\forall x \in R^N$ where $\alpha, \beta > 0$ are constants (recall $m_+[x, y] = \lim_{h \to 0+} \frac{\|x+hy\| - \|x\|}{h}$)

(a) Show $\exists M > 0$ such that $\|z\| \leq M \Rightarrow \|S(t)z\| \leq M$, $\forall t \geq 0$.

(b) Show (3) has a τ periodic solution u_{z_τ} where $\|z_\tau\| \leq M$ for each $\tau > 0$.

(c) Show $\exists w \in R^N$, $\|w\| \leq M$, such that $f(w) = \theta$.

MA 778 – 779, Measure Theory and Probability **PhD. Qualifying Exam**

Solve any four of the following five problems. If you are quoting a major theorem, check that the assumptions are satisfied.

1. Let $\mathcal{A}$ be a collection of subsets of Ω. $\mathcal{A}$ is said to be a π-system if $\mathcal{A}$ is closed under finite intersections. $\mathcal{A}$ is said to be a λ-system if:

a) $\Omega \in \mathcal{A}$,
b) $\mathcal{A}$ is closed under finite disjoint unions,
c) if $A \in \mathcal{A}$, $B \in \mathcal{A}$, and $B \subset A$, then $A - B \in \mathcal{A}$ ($A - B = A \cap B^c$), and
d) if $\{A_n\}$ is an increasing sequence of sets in $\mathcal{A}$, then $\lim_{n\to\infty} A_n \in \mathcal{A}$.

Prove that $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is both a π-system and a λ-system.

2. For random variable X, $k > 1$ and positive α, show that $E[X] = 0$ and $E[|X|^k] < \infty$ imply

$$\lim_{n\to\infty} n^{\alpha(k-1)} E[X 1_{(|X| \le n^\alpha)}] = 0.$$

3. (i) State Fubini's theorem and Jensen's inequality.
(ii) Use Fubini's theorem and Jensen's inequality to show, if random variables X and Y are independent, $E[|X|^p] < \infty$ for some $p \ge 1$, and $E[Y] = 0$, then $E[|X+Y|^p] \ge E[|X|^p]$.

4. Let f be a continuous function on $[0,1]$. Use the Strong Law of Large Numbers to show that for any $x \in [0,1]$

$$\lim_{n\to\infty} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = f(x).$$

5. (i) Give the definition of conditional expectation.
(ii) Suppose for random variables X and Y with finite second moments $E[X|Y] = Y$ a.s. and $E[Y|X] = X$ a.s. Show that $X = Y$ a.s.