Proposition de stage de MASTER 2 par Luc Molinet (LMPT) :
“Scattering for the H^1-critical Schrödinger equation ”.

The aim of this stage is the study of the paper *”Global well-posedness, scattering and blow-up for the energy, focussing, nonlinear Schrödinger equation in the radial case”* by C. Kenig and F. Merle. The strategy developed in this paper is in the heart of a lot new researches in this field.

In this paper the author are interesting in the H^1-critical focussing nonlinear Schrödinger equation in $\mathbb{R}^N$, $N = 3, 4, 5$:

$$(NLS) \begin{cases} i\partial_t u + \triangle u + |u|^{\frac{4}{N-2}} u = 0 \\ u(0) = u_0 \in \dot{H}^1(\mathbb{R}^N) \end{cases}$$

This equation is hamiltonian and thus has got a conserved energy :

$$E(v) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 - \frac{N-2}{2N} \int_{\mathbb{R}^N} |v|^{\frac{2N}{N-2}} \; .$$

From well-posedness theory it is known that solutions with small $\dot{H}^1(\mathbb{R}^N)$ initial data exist for all time and scatter at $+\infty$ and -∞, i.e. there exist $u_{0,+}, u_{0,-}$ in $\dot{H}^1$ such that

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta} u_{0,+}\|_{\dot{H}^1} = 0, \text{ and } \lim_{t\to-\infty} \|u(t) - e^{it\Delta} u_{0,-}\|_{\dot{H}^1} = 0$$

Note that the smallness assumption in $\dot{H}^1(\mathbb{R}^N)$ forces the energy to be positive by Sobolev embedding. On the other hand from a classical virial identity it is well-known that some solution emanating from initial data with a negative energy must break down in finite time. Moreover,

$$W(x) = W(x,t) = \Big(1 + \frac{|x|^2}{N(N-2)}\Big)^{\frac{2-N}{2}}$$

is in $\dot{H}^1(\mathbb{R}^N)$ and is a stationary solution of (NLS). Therefore, scattering cannot always occur even for global in time solutions.

The aim of this paper is to initiate the detailed classification of the solutions to (NLS) by proving that any solutions with initial data u_0 satisfying : $E(u_0) < E(W)$ and $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ are global in time and scatter at $+\infty$ and $-\infty$.

This paper is mainly self-contained but a mininum background in PDE’s is required.

References

[1] T. Cazenave and F.B. Weissler, *The Cauchy problem for the critical nonlinear Schr ödinger equation in H^s* . Nonlinear Anal., Theory Methods Appl. 14, 807836 (1990)

[2] T. Hmidi and S. Keraani, *Blowup theory for the critical nonlinear Schr ödinger equations revisited.* Int. Math. Res. Not. 2005, no. 46, 2815–2828.

[3] C.E. Kenig and F. Merle, Frank, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schr ödinger equation in the radial case.* Invent. Math. 166 (2006), no. 3, 645–675.