Exact rounding of the elementary functions
and Diophantine approximation

The *Arénaire project*
http://www.ens-lyon.fr/LIP/Arenaire/index.html.en
aims at elaborating and consolidating knowledge in the field of Computer Arithmetic. Reliability, accuracy, and speed are the major goals that drive our studies. It contributes to the improvement of the available arithmetic, at the hardware level as well as at the software and algorithmic levels, on computers, processors, dedicated or embedded chips, etc. Improving computing does not necessarily mean getting more accurate results or getting them more quickly: one also takes into account other constraints such as power consumption, or the reliability of numerical software.

One of the four main directions is *Models and properties of computation, validation and proof — C++, Coq, PVS* including *Correct rounding, interval arithmetic, formal proof.*

The question of exact rounding of the elementary functions involves a result from Diophantine approximation which is a quantitative refinement of Hermite–Lindemann Theorem. This theorem asserts that *for any non-zero complex number z, one at least of the two numbers z, e^z is transcendental* – and therefore is irrational. As a consequence, if a and b are rational numbers with $b \neq 0$, then $e^b \neq a$. The question is to give a lower bound for $|e^b - a|$. In the case where a and b are positive integers, this question has been investigated by K. Mahler (1953, 1967), M. Mignotte (1974) and F. Wielonsky (1997) who proved

$$|e^b - a| > e^{-20(\log a)(\log b)}.$$

In the case where a and b are rational numbers (and more generally algebraic numbers), such a lower bound follows from Gel'fond–Baker's method.

For $p/q \in \mathbf{Q}$ with $q > 0$ and $\gcd\{p, q\} = 1$, define $H(p/q) = \max\{|p|, q\}$.

Theorem. *There exists an absolute constant A_0 such that, for any a and b in $\mathbf{Q}$ with $b \neq 0$,*

$$|e^b - a| \geq \exp\{-1.3 \cdot 10^5 (\log A)(\log B)\}$$

where $A = \max\{H(a),\ A_0\}$, $B = \max\{H(b),\ 2\}$.

The purpose of the work is to study applications of this estimate in theoretical computer science.

Reference

Muller, J-M.; Tisserand, A. – Towards exact rounding of the elementary functions. Alefeld, Goetz (ed.) et al., *Scientific computing and validated numerics.* Proceedings of the international symposium on scientific computing, computer arithmetic and validated numerics SCAN-95, Wuppertal, Germany, September 26-29, 1995. Berlin: Akademie Verlag. Math. Res. 90, 59-71 (1996).

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