University of Orléans **Master 2**
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Project of master thesis

Title : *Dispersive PDE's*
Subtitle : *(Non)linear Schrödinger and wave equations*

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Nonlinear evolution equations have been intensively studied during the past twenty years. This project is concerned with the most classical examples, namely the Schrödinger equation

$$\begin{cases} \partial_t u(t,x) - \Delta_x u(t,x) = v(t,x) \\ u(0,x) = f(x) \end{cases} \tag{1}$$

and the wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = v(t,x) \\ u(0,x) = f_0(x),\ \partial_t\big|_{t=0} u(t,x) = f_1(x) \end{cases} \tag{2}$$

in the Euclidean space $\mathbb{R}^n$, with polynomial type nonlinearities :

$$v(t,x) = |u(t,x)|^{\gamma} \quad \text{or} \quad \pm\ u(t,x)\,|u(t,x)|^{\gamma-1}\,.$$

A general strategy has been developped in order to study such equations. Let us briefly summarize it for the Schrödinger equation (1). The first step consists in establishing dispersive estimates

$$\|u(t,x)\|_{L_x^{q'}} \leq C\,|t|^{-n(\frac{1}{2}-\frac{1}{q})}\,\|f\|_{L_x^q}$$

for the linear homogeneous equation

$$\begin{cases} \partial_t u(t,x) - \Delta_x u(t,x) = 0\,, \\ u(0,x) = f(x)\,. \end{cases}$$

Here $2 \leq q \leq \infty$ and $1 \leq q' \leq 2$ are conjugate indices. The second step consists in deducing Strichartz type estimates

$$\|u(t,x)\|_{L_t^{\widetilde{p}'} L_x^{\widetilde{q}'}} \leq C\,\big\{\,\|f\|_{L_x^2} + \|v(t,x)\|_{L_t^p L_x^q}\,\big\}$$

for the linear inhomogeneous equation

$$\begin{cases} \partial_t u(t,x) - \Delta_x u(t,x) = v(t,x)\,, \\ u(0,x) = f(x)\,. \end{cases}$$

Here the indices $2 \leq p, \widetilde{p} \leq \infty$ and $2 \leq q, \widetilde{q} < \infty$ satisfy $\frac{2}{n}\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and $\frac{2}{n}\frac{1}{\widetilde{p}}+\frac{1}{\widetilde{q}}=\frac{1}{2}$. The third step consists in applying these estimates to the nonlinear equation (1) and getting for instance wellposedness (existence and uniqueness of solutions). The strategy is similar for the wave equation (2) but more involved. For instance, Lebesgue spaces L^q_x are replaced by homogeneous Sobolev or Besov spaces.

The first aim of the project consists in understanding this strategy and the tools involved (a beautiful blend of harmonic analysis and PDE's), first for the Schrödinger equation and next for the wave equation. The second aim consists in clarifying some technical points, such as the Christ–Kiselev lemma, the Keel–Tao endpoint, or the minimal regularity assumptions required for the wave equation. Further issues include scattering theory and the Morawetz inequality.

References

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