University of Orléans **Master 2**
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Project of master thesis

Title: *The Hardy space H^1*
Subtitle: *Atomic decompositions & bounded operators*

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In Fourier analysis on Euclidean space, results often hold for the whole scale of Lebesgue spaces L^p $(1<p<\infty)$, but fail in the limit case $p=1$ (and $p=\infty$). A typical example is the L^p boundedness of singular integral operators such as the Hilbert transform in dimension 1 and Riesz transforms in higher dimensions.

The Hardy space H^1 has proved to be a proper substitute for L^1, and its dual BMO for L^∞. For instance, singular integral operators are bounded from H^1 to L^1 and by duality from L^∞ to BMO.

In the seventies, Hardy space theory was greatly simplified with the discovery of atomic decompositions (see [CW], [S]). Recall that an *atom* is a measurable function a which satisfies the following conditions: a is supported in a ball B, $\int_B a = 0$ and $\|a\|_{L^\infty} \leq |B|^{-1}$. Then

$$H^1 = \left\{ f = \textstyle\sum_j \lambda_j a_j \;\middle|\; a_j \text{ atoms}, \sum_j |\lambda_j| < +\infty \right\}$$

and the norm $\|f\|_{H^1}$ is the infimum of $\sum_j |\lambda_j|$ taken over all atomic decompositions of f.

In order for a linear operator T to be bounded from H^1 to L^1, it is necessary that

$$(\star) \qquad \sup \left\{ \|Ta\|_{L^1} \;\middle|\; a \text{ atom} \right\} < \infty\,.$$

For a long time, this condition was commonly believed to be sufficient, until the recent discovery by Bownik [B] of a counterexample. This result came as a surprise, although the idea goes back actually to Y. Meyer.

Subsequently Meda, Sjögren & Vallarino [MSV] found a sufficient condition. Instead of standard atoms, they take the supremum in $(\star)$ over $(1,q)$–*atoms* i.e. measurable functions a supported in balls B such that $\int_B a = 0$ and $\|a\|_{L^q} \leq |B|^{-1/q'}$. Here $1<q<\infty$ is fixed and q' denotes the dual index. The proof relies on maximal characterizations of H^1.

The project consists in understanding the Hardy space H^1 at the light of the recent works [B] and [MSV].

References

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