Study of Capillary Equilibrium Surfaces

Master Thesis Proposal

Location : CEA Saclay

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Duration : 4 to 6 months

Financial conditions : 700 euros/month gross (around 550 euros/month net), possible accomodation allowance around 220 euros/month and a bonus at the end of the internship.

We propose through this Master Thesis subject a rough review of some problems related to the surface Γ that separates two fluids at mechanical equilibrium. Both fluids are supposed in thermal equilibrium and we shall only take capillary effects, gravity and the influence of solid boundaries into account.

The dimension of the ambient space is (d+1), and we consider both fluids to be enclosed within a portion of space delimited by $\Omega \times I$, where Ω is a bounded open subset of $\mathbb{R}^d$ and I is an open set of $\mathbb{R}$.

1 The Minimal Surface Problem

A first historical approach to this problem has led to a pure geometrical called the Minimal Surface Problem. This problem consist in determining Γ as a curve with a minimal mean curvature under additional constraints prescribed by the physical context (gravity, solid boundaries for example). If one seek Γ as the graph of a fonction $u : \Omega \to \mathbb{R}$, then u is meant to verify the following famous variational problem:

$$J(u) = \min_{v \in \mathrm{BV}(\Omega)} J(v), \text{ where} \tag{1}$$

$$J : v \in \mathrm{BV}(\Omega) \mapsto \int_\Omega \frac{\mathrm{d}x}{\sqrt{1+|\nabla u|^2}} + \frac{\kappa}{2}\int_\Omega v^2 \mathrm{d}x + \int_{\partial\Omega} \beta v \mathrm{d}\gamma, \tag{2}$$

with $\kappa \in \mathbb{R}_+$, and $\beta : \partial\Omega \to \mathbb{R}$ such that $\beta \leq 1$. If u is a regular solution of (1)-(2), then u is also a solution of the well-known problem:

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + \kappa u = 0 \quad \text{in } \Omega, \tag{3}$$

$$-\frac{\mathbf{n}\cdot\nabla u}{\sqrt{1+|\nabla u|^2}} = \beta \quad \text{on } \partial\Omega, \tag{4}$$

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where $\mathbf{n}$ is the outter unit normal to Ω.

We propose to elaborate a short review of such problems using two articles. The first one [2] considers a simplified symmetric problem and seek for the solution as the stationnary state of an evolutionnary equation. The second article [1] examines the general cases with additional constraints such as a volume constraint on one of the fluid. This article also refers to a serie of other paper by the same author about the same topic.

The student is not intended to acquire an exhaustive knowledge of this famous problem, but she/he should be able to grasp essential notions related to it. Both article shall be considered as initial lead.

2 Diffuse Interface Modelling

The following work is considered as a complementary work to bibliographical task of the first section. An alernative to the surface minimal problem consist in considering the interface that separates both fluids as non-zero thick transition zone. Such approach is referred as diffused interface models and is often used in Computational Physics. We also propose to examine the possible connection between a diffuse interface model at equilibrium and the minimal surface problem. One may for example consider the case of a simple 2D bubble at equilibrium with its surrounding neglecting gravity effects and try to recover the Laplace equation that connects the pressure jump across the material interface and the curvature of the bubble.

We propose for such problem to examine stationnary solutions of an isothermal Euler-type model such as

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \tag{5}$$

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \times \mathbf{u} + (P - \sigma|\nabla\alpha|^2/2)\mathrm{Id} + \sigma\nabla\alpha \otimes \alpha) = \rho \mathbf{g}, \tag{6}$$

$$\partial_t \alpha + u \cdot \nabla\alpha = 0, \tag{7}$$

where α is a smooth function valued in $[0, 1]$ such that $\alpha = 1$ (resp. 0) in fluid 1 (resp. 0).

3 Numerical Tests

The following is also considered as a complementary work. The student may also perform numerical simulations using programming tools of her/his choice (C/Fortran/Matlab/Scilab/Python…) in order to compute equilibrium capillary surfaces. This may involve minimal surfaces computation or the simulation of diffuse interface models.

References

[1] C. GERHARDT. *A Free Boundary Valued Problem for Capillary Surfaces*, Pacific Journal of Math., vol. 88, No. 2, 1980.

[2] N. ISHIMURA. *Existence of Symmetric Capillary Surfaces via Curvature Evolution*, J. Fac. Sci. Univ. Tokyo. Sect. IA, Math., No. 40, 1993.