Subject: Uniform controllability of controlled partial differential equations.

Consider an infinite dimensional linear control system

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0, \tag{1}$$

where the state $y(t)$ belongs to a Hilbert space X, the control $u(t)$ belongs to a Hilbert space U, $A : D(A) \to X$ is an operator, and B is a control operator (in general, unbounded) on U. Discretizing this partial differential equation, using for instance a finite difference, or a finite element scheme, leads to a family of finite dimensional linear control systems

$$\dot{y}_h(t) = A_h y_h(t) + B_h u_h(t), \quad y_h(0) = y_{0h}, \tag{2}$$

where $y_h(t) \in X_h$ and $u_h(t) \in U_h$, for $0 < h < h_0$.

Let $y_1 \in X$; if the control system (1) is controllable in time T, then there exists a solution $y(\cdot)$ of (1), associated with a control u, such that $y(T) = y_1$. The following question arises naturally: is it possible to find controls u_h, for $0 < h < h_0$, converging to the control u as the mesh size h of the discretization process tends to zero, and such that the associated trajectories $y_h(\cdot)$, solutions of (2), converge to $y(\cdot)$? Moreover, does there exist an efficient algorithmic way to determine the controls u_h?

For controllable linear control systems of the type (1), we have available many methods in order to realize the controllability. A well known method, adapted to numerical implementations, is the Hilbert Uniqueness Method (HUM), introduced in [7], which consists in minimizing a cost function, namely, the L^2 norm of the control. We can investigate the above question in the case where controllability of (1) is achieved using the HUM method, and the objective is to establish conditions ensuring a *uniform controllability property* of the family of discretized control systems (2), and to establish a computationally feasible approximation method for realizing controllability.

The question of uniform controllability and/or observability of the family of approximation control systems (2) has been investigated by E. Zuazua and collaborators in a series of articles [1, 4, 10, 11, 12], for different discretization processes, on different examples. When the observability constant of the finite dimensional approximation systems does not depend on h, one says that the property of *uniform observability* holds. For classical finite difference schemes, a uniform boundary observability property does not hold for 1-D wave equations [4]. It is actually well known that discretization processes generate high frequency spurious solutions that do not exist in the continuous problem, and that may lead, in the exact controllability problem, to the divergence of the control approximations.

In the parabolic case, the strong dissipative properties induced by the analyticity of the semigroup totally damp out these spurious high frequencies in 1-D but do not suffice however in general in the multi-dimensional case; a general uniform controllability property of the discretized models (2) was established in [5] in the case where the operator A generates an analytic semigroup, under standard assumptions on the discretization process. A minimization procedure to compute the approximation controls was provided, and an important fact in view of the numerical implementation is that the uniform property implies a uniform conditioning of the gradient method.

In the hyperbolic case, where there is no such strong damping, the divergence phenomenon of controls may be drastic (see [4, 12]). Recent results [2] show that a general answer must take into account the choice of a discretization scheme.

The objective of this subject is to investigate the problem of uniform controllability for controlled PDE's. The main references are [5, 12].

References

[1] C. Castro, S. Micu, Boundary controllability of a linear semi-discrete 1-D wave equation derived from a mixed finite element method, Numer. Math.

[2] S. Ervedoza, C. Zheng, E. Zuazua, *On the observability of time-discrete conservative linear systems*, J. Functional Analysis bf 254 (2008), no. 12, 3037–3078.

[3] R. Glowinski, J.-L. Lions, Exact and approximate controllability of distributed parameter systems, Acta Numerica (1996) 159-333.

[4] J.A. Infante, E. Zuazua, Boundary observability for the space semi-discretizations of the 1-D wave equation, M2AN Math. Model. Numer. Anal. 33, 2 (1999) 407-438.

[5] S. Labbé, E. Trélat, *Uniform controllability of semidiscrete approximations of parabolic control systems*, Syst. Cont. Letters **55** (2006), no. 7, 597–609.

[6] I. Lasiecka, R. Triggiani, Control theory for partial differential equations: continuous and approximation theories. I. Abstract parabolic systems, Encyclopedia of Mathematics and its Applications, 74, Cambridge University Press, Cambridge, 2000.

[7] J.-L. Lions, Exact controllability, stabilization and perturbations for distributed systems, SIAM Rev. 30 (1988) 1-68.

[8] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Applied Math. Sci. 44, Springer-Verlag, 1983.

[9] D.L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions, SIAM Rev. 20, 4 (1978) 639-739.

[10] E. Zuazua, Controllability of partial differential equations and its semi-discrete approximations, Discrete Contin. Dyn. Syst. 8, 2 (2002) 469-513.

[11] E. Zuazua, Optimal and approximate control of finite-difference approximation schemes for the 1-D wave equation, Rendiconti di Matematica VIII, 24, II (2004) 201-237.

[12] E. Zuazua, Propagation, Observation, Control and Numerical Approximation of Waves approximated by finite difference method, SIAM Review 47, 2 (2005) 197-243.