Master PUF Orléans-Ho Chi Minh Year 2007/2008

Examination of Mathematical Finance

(Three hours : authorized lecture notes)

The two exercices are independent. Please provide separate copies for each exercice.

Exercice 1

Given a Brownian motion $(B_t)_{t\geq 0}$ with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$, we consider the SDE

$$\forall t \geq 0,\ dX_t = b(X_t)dt + dB_t, \quad X_0 = 0, \tag{1}$$

where $b(x) = \exp(-|x|^{1/2})$, $x \in \mathbb{R}$. In what follows, ϕ stands for the primitive of b vanishing at 0:

$$\forall x \in \mathbb{R},\ \phi(x) = \int_0^x b(u)du.$$

It is well seen that ϕ is bounded. With these notations at hand, we put

$$\forall x \in \mathbb{R},\ s(x) = \exp(-2\phi(x)), \quad S(x) = \int_0^x s(u)du.$$

It is clear that the function S is increasing of class $\mathcal{C}^2$ from $\mathbb{R}$ onto itself. The function S admits an inverse, denoted by S^{-1}, of class $\mathcal{C}^2$ from $\mathbb{R}$ onto itself (i.e. $S^{-1} \circ S(x) = x$ and $S \circ S^{-1}(x) = x$ for all $x \in \mathbb{R}$).

1. Is it possible to apply the existence and uniqueness theorem to (1) ? Why ?
2. In this question, we assume that (1) admits a solution X in M^2 i.e. $\mathbb{E}\int_0^t X_s^2 ds < +\infty$, for all $t \geq 0$. Setting $Y_t = S(X_t)$ for $t \geq 0$, show that

$$\forall t \geq 0,\ dY_t = g(Y_t)dB_t, \quad Y_0 = 0,$$

   where $g(x) = s \circ S^{-1}(x)$ for all $x \in \mathbb{R}$.
3. Show that g is differentiable and that there exists a constant $K \geq 0$ such that

$$\forall x \in \mathbb{R},\ |g'(x)| \leq K.$$

4. Show that the SDE

$$\forall t \geq 0,\ dZ_t = g(Z_t)dB_t, \quad Z_0 = 0,$$

   admits a unique solution (in M^2).
5. Deduce that (1) admits at most one solution X in M^2.
6. Conversely, prove that $(S^{-1}(Z_t))_{t\geq 0}$ belongs to M^2 and satisfies (1).
7. What can you say about (1) ?

Exercice 2 : Asian Option

We consider the Black-Scholes model :

$$dS_t \;=\; rS_t dt + \sigma S_t d\hat{W}_t,$$

where r is the constant interest rate, and $\hat{W}$ is a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ under the risk-neutral probability measure $\mathbb{Q}$. We are interested in the Asian option with payoff at maturity T :

$$H_T \;=\; \Big(\frac{1}{T}\int_0^T S_t dt - K\Big)_+,$$

and we recall that the time t price of this option is given by :

$$V_t \;=\; \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)}\Big(\frac{1}{T}\int_0^T S_t dt - K\Big)_+\Big|\mathcal{F}_t\Big].$$

1) Give the explicit form of S_t under $\mathbb{Q}$.

2) We denote $\tilde{S}_t = e^{-rt}S_t$ the discounted stock price. Prove that

$$\mathbb{E}^{\mathbb{Q}}\Big[e^{-rT}(S_T - K)_+|\mathcal{F}_t\Big] \;\geq\; (\tilde{S}_t - Ke^{-rT})_+$$

and deduce that

$$\mathbb{E}^{\mathbb{Q}}\Big[(\tilde{S}_t - Ke^{-rT})_+\Big] \;\leq\; \mathbb{E}^{\mathbb{Q}}\Big[e^{-rT}(S_T - K)_+\Big], \quad 0 \leq t \leq T. \tag{2}$$

3) We admit as a consequence of Jensen's inequality that for any convex function $f : \mathbb{R} \to \mathbb{R}$, and for any continuous function $g : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$, we have

$$f\Big(\frac{1}{T}\int_0^T g(t, S_t)dt\Big) \;\leq\; \frac{1}{T}\int_0^T f(g(t, S_t))dt. \tag{3}$$

By integrating between $t = 0$ and $t = T$ each of the two terms in (2), and by applying appropriately the convex inequality (3), deduce that

$$V_0 \;\leq\; \mathbb{E}^{\mathbb{Q}}\Big[e^{-rT}(S_T - K)_+\Big],$$

i.e. the Asian option price is smaller than the European call option price.

4) Let (Y_t) be the process defined by

$$Y_t \;=\; \frac{1}{S_t}\Big(\frac{1}{T}\int_0^t S_s ds - K\Big).$$

a) Prove that (Y_t) is solution to the stochastic differential equation

$$dY_t \;=\; \Big(\frac{1}{T} + (\sigma^2 - r)Y_t\Big)dt - \sigma Y_t d\hat{W}_t.$$

b) Show that

$$V_t \;=\; e^{-r(T-t)}S_t\mathbb{E}^{\mathbb{Q}}\Big[\Big(Y_t + \frac{1}{T}\int_t^T S_s^t ds\Big)_+\Big]$$

with

$$S_s^t = \exp\Big((r - \frac{\sigma^2}{2})(s-t) + \sigma(\hat{W}_s - \hat{W}_t)\Big), \quad t \leq s.$$

c) Deduce that $V_t = e^{-r(T-t)} S_t F(t, Y_t)$ where the function F is defined by

$$F(t, y) = E^{\mathbb{Q}}\Big[\Big(y + \frac{1}{T}\int_t^T S_s^t ds\Big)_+\Big].$$

d) By assuming that F is smooth C^2, and by using Itô's formula, determine a perfect hedging portfolio strategy for the Asian option, that one will express in terms of S_t, Y_t, and the derivatives of F.

5) We set

$$\hat{V}_0 = \mathbb{E}^{\mathbb{Q}}\Big[e^{-rT}\Big(\exp\big(\frac{1}{T}\int_0^T \ln S_t dt\big) - K\Big)_+\Big].$$

a) By applying appropriately the convex inequality (3), prove that $V_0 \geq \hat{V}_0$.

b) We admit that the random variable $I_T = \int_0^T \hat{W}_t dt$ is normally distributed under $\mathbb{Q}$. Show that its mean is zero and its variance is equal to $T^3/3$ under $\mathbb{Q}$. *Indication* : to calculate the variance, apply Itô's formula to I_t^2 where $I_t = \int_0^t \hat{W}_u du$.

c) Deduce that

$$\hat{V}_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}\Big[\Big(S_0 \exp\Big((r - \frac{\sigma^2}{2})\frac{T}{2} + \sigma\sqrt{\frac{T}{3}}U\Big) - K\Big)_+\Big],$$

where U is a centered reduced normal random variable under $\mathbb{Q}$.

d) Give an explicit form for $\hat{V}_0$ in terms of the cumulated distribution function of the centered reduced normal random variable.