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De thi 2 khoa cuoi khoa lop hoc PUF
(mon hoc: Mo hinh hoa Giai tich so)

Bai tap 1

Cho $\Omega = (0,1)$. Ta dat $L^2(\Omega) = L2,\ H^1(\Omega) = H1$.

1. a) Cho $v \in H^1(\Omega)$, $v(\frac{1}{2})$ co nghia hay khong?

b) Chung minh cai tap hop $V = \{v \in H^1(\Omega),\ v(\frac{1}{2}\}$ la tap hop dong trong $H^1(\Omega)$.

2. Chung minh co ton tai mot hang so $C > 0$ sao cho $||v||_{L2} \leq C||v'||_{L2}$, $\forall v \in V$.

3. a) Xay mot day ham so $v_n \in H^1(\Omega)$ sao cho $||v_n||_{L2} \to +\infty$, khi $n \to +\infty$ voi $||v'_n||_{L2} \leq \widetilde{C}$, $\widetilde{C}$ hang so doc lap doi voi n.

b) Ket qua chot nay co mau thuan gi voi bat dang thuc cua Poincare hay khong ?

Bai tap 2

Cho $\Omega = (0,1)$. Ta dat $L^2(\Omega) = L2,\ H^1(\Omega) = H1$.

Da cho 2 ham so $c \in C(\overline{\Omega})$,$f \in L2$. Ta gia su rang co ton tai mot hang so $C_0 > 0 \mid c(x) \geq C_0$, $\forall x \in [0,1]$.

Ta xet bai toan bien phan duoi day

$-u"(x) + c(x)u(x) = f(x), \quad x \in \Omega$

(1) $\quad u(0) = u(1);\ \ u'(1) = u'(0) + \beta$

β hang so da cho truoc.

1. Chung minh cai tap hop $V = \{v \in H^1(\Omega)\ |v(1) = v(0)\}$ la mot tap hop Hilbert voi mot tich vo huong ma ta se chinh xac.

2. Lap cong thuc bien phan cua bai toan (1) duoi hinh thuc

(2) $\quad a(u,v) = L(v),\ u, v \in V$

voi L la mot dang tuyen tinh lien tuc tren V va a mot dang song tuyen tinh lien tuc khang tu tren V. Ton tai duy nhat nghiem u cua bai toan (2).

3. Nguoc lai chung minh la moi u loi giai cua (2) la mot nghiem cua (1) theo mot y nghia ma ta se chinh xac vao.

4. Chung minh co ton tai mot hang so $C > 0$ sao cho

$||u||_{H1} \ \leq C(||f||_{L2} + |\beta|)$

Hay ket luan.

5. Ta gia su rang $c(x) = 1$.

Cho $h = 1/N$, N so nguyen > 0. Dat $x_i = ih$, $0 \leq i \leq N$. Goi V_h la khong gian cua nhung ham $v_h : \overline{\Omega} \longrightarrow \mathbf{R}$, v_h lien tuc va affin tren moi khoang $[x_i, x_{i+1}]$ va thoa dung $v_h(0) = v_h(1)$.

a) Viet mot co so chinh tac cua V_h va tinh chieu cua V_h.

b) Viet chi tiet mot he thong tuyen tinh cho phep ta xac dinh mot loi giai gan dung cua nghiem cua bai toan (2).

c) Tu do tinh toan ma tran cung A_h cua he thong gan dung trong cau hoi b).

Bai mau cua de thi lop hoc Puf (A. Pham Ngoc Dinh)

1. Xet ham $\psi(x) = \int_0^x [\varphi(\xi) - \theta(\xi)\int_0^1 \varphi(t)dt]d\xi$, ta co $\psi" = \varphi' - \theta' \int_0^1 \varphi(t)dt$. Tu dinh nghia cua $\widetilde{V}$ ta co $\int_0^1 v\psi" = \int_0^1 g\psi$ voi ψ va $\psi"$ viet phia tren. Ta co the viet lai cai bieu thuc cua $\int_0^1 v\varphi'$dung Fubini:

(1) $\int_0^1 v\varphi' = \int_0^1 [\int_\xi^1 g(x)dx]\varphi(\xi)d\xi + \int_0^1 \varphi(t)dt[\int_0^1 v\theta' dx - \int_0^1\int_\xi^1 g(x)\theta(\xi)d\xi dx]$

$= \int_0^1 [\int_\xi^1 g(x)dx + C]\varphi(\xi)d\xi$, voi $C = \int_0^1 v\theta' dx - \int_0^1 \int_\xi^1 g(x)\theta(\xi)d\xi dx$.

Tu cong thuc (1) suy ra

(2) $v' = -[\int_\xi^1 g(x)dx + C]$

tuc la $v' \in C^0([0,1])$. Vay $v" = g$ hh Ω. Cuoi cung $v \in H^2(\Omega)$, $\Omega = (0,1)$.

Nguoc lai neu $v \in H^2(\Omega)$ thi ta co $\int_0^1 v\varphi" = -\int_0^1 v'\varphi' = \int_0^1 v"\varphi$. Vay thi lay $g = v" \in L^2(\Omega)$ dan den $v \in \widetilde{V}$.

2. Xet phiem ham $J : \mathbf{R}^m \to \mathbf{R}$ lien tuc sao cho $\lim J(v) = +\infty$ khi $||v||_{\mathbf{R}^m} \to +\infty$. Gia thiet ve J keo theo $\inf_{v\in\mathbf{R}^m} J(v) = \inf_{||v||\leq\alpha} J(v) = \beta$, voi α, β huu han.

Cho $\{w_j\}$ mot day cuc tieu hoa tuc la $J(w_j) \to \beta$ khi $j \to \infty$ voi $||w_j||_{\mathbf{R}^m} \leq \alpha$. Do Bolzano-Weierstrass co ton tai mot day con $\{w_\mu\} \subset \{w_j\}$ va $u \in \mathbf{R}^m$ sao cho $||w_\mu - u||_{\mathbf{R}^m} \to 0$ khi $\mu \to \infty$. Tinh lien tuc cua J dan den $J(u) = \beta$.

3. a) Cho tap $\{\varphi_j\}$, $\varphi_j(x) = \sqrt{2}\sin(j\pi x)$ mot co so truc chuan cua $L^2(\Omega)$. O day ta co $\varphi_j" = -(j\pi)^2\varphi_j$.

Xet $V = Vect\{\varphi_j\} \subset H_0^1 \cap H^2 = \overline{V} \oplus \overline{V}^\perp$. Tren $H_0^1 \cap H^2$ ta de tich vo huong $< u, v >= \int_0^1 (u'v' + u"v")dx$ voi $||u|| = \sqrt{< u, u >}$.

Cho $v \in V^\perp$. ta co $\int_0^1 (\varphi_j' v' + \varphi_j" v")dx = 0$. Cai bieu dang thuc chot nay dan den $(-(j\pi)^2 - 1)\int_0^1 \varphi_j v" dx = 0$ do phuong trinh vi phan duoc ham φ_j thoa dung. Vay cuoi cung ta duoc $\int_0^1 \varphi_j v" dx = 0$, $\forall j$, tuc la $v"(x) = 0$. Cho nen $v(x) = \alpha x + \beta$ va nhung dieu kien bien $v(0) = v(1) = 0$ dan den $v = 0$. Vay $V^\perp = \{0\}$ va do do $H_0^1 \cap H^2 = \overline{V}$.

Mat khac $\int_0^1 \varphi_i\varphi_j dx = 0$ vi the $< \varphi_i, \varphi_j >= 0$ tuc la ham $\psi_j = \varphi_j/||\varphi_j||$ la mot co so truc chuan cua $H_0^1 \cap H^2$.

b) Xet phiem ham J xac dinh boi $J(v) = \frac{1}{2}\int_0^1 v'^2 dx - \int_0^1 \int_0^{v(x)} f(x,t)dtdx$ va cho V_m la khong gian m chieu sinh boi $\{\varphi_1, ..., \varphi_m\}$.

Ta co $\int_0^1 |v|dx \leq \sqrt{\int_0^1 v^2 dx} \leq ||v'||_0$ (bat dang thuc Poincare) voi $||.||_0 = ||.||_{L2(\Omega)}$.

Dat $M = \max_{(x,t)\in\Omega\times\mathbf{R}} f(x,t)$, ta co $|\int_0^1 dx \int_0^{v(x)} f(x,t)dt| \leq M\int_0^1 |v|dx \leq ||v'||_0$. Vay thi

$J(v_m) \geq \frac{1}{2}||v_m'||_0^2 - M||v_m'||_0 \to \infty$ khi $||v_m'||_0 \to \infty$ vi o chieu huu han tat ca chuan tuong duong voi nhau. Vi vay co ton tai $u_m \in V_m$ sao cho $J(u_m) = \min_{v\in Vm} J(v)$.

Ta tinh bay gio $J'(v)$ dao ham Gateaux cua J.

Dat $\Delta = \dfrac{J(v+\lambda\varphi) - J(v)}{\lambda} = \Delta_1 + \Delta_2$

voi $\Delta_1 = \frac{1}{2}\int_0^1 \dfrac{(v' + \lambda\varphi')^2 - v'^2}{\lambda}$, $\Delta_2 = -\frac{1}{\lambda}\Big[\int_0^1\int_0^{v+\lambda\varphi} f - \int_0^1\int_0^v f\Big]$.

Ro rang ta co $\lim_{\lambda\to o^+}\Delta_1 = \int_0^1 v'(x)\varphi'(x)dx$ va $\lim_{\lambda\to o^+}\Delta_2 =_- \int_0^1 \varphi(x)f(x,v(x))dx$ vi Δ_2 cung con the viet duoi hinh thuc $\Delta_2 =_- \int_0^1 \varphi(x)f(x,v(x)+\theta\lambda v(x))dx$ voi $0<\theta<1$. Vay cuoi cung ta co

(3) $<< J'(u_m),\varphi >>= \int_0^1 u_m'(x)\varphi'(x)dx - \int_0^1 f(x,u_m(x))\varphi(x)dx \ \forall\varphi\in V_m$.

Vi ta co cuc tieu hoa tren ca V_m ham u_m thoa he thuc

(4) $\int_0^1 u_m'(x)\varphi_j'(x)dx = \int_0^1 f(x,u_m(x))\varphi_j(x)dx,\ \forall j=1,...,m$

c) Ta co $\frac{1}{2}||u_m'||_0^2 - M||u_m'||_0 \le J(u_m) \le J(0) = 0$. Cho nen $||u_m'||_0 \le 2M$ tuc la day $\{u_m\}$ nam trong mot tap hop bi chan cua H_0^1, tu do co ton tai mot day con $\{u_\mu\}\subset\{u_m\}$ va $u\in H_0^1$ sao cho $u_m\to u$ trong H_0^1 yeu. Phep nhung $H_0^1\subset C^0(\overline{\Omega})$ compac dan den $u_m\to u$ trong $C^0(\overline{\Omega})$.

Cho j co dinh $<m$. Ta co $-\int_0^1 u_m'(x)v'dx + \int_0^1 f(x,u_m(x))dx = 0$ keo theo

(5) $\int_0^1 u_m(x)v"dx + \int_0^1 f(x,u_m(x))v(x)dx = 0\ \forall v\in V_j$.

Ap dung dinh lebesgue (f lien tuc va bi chan) ta co $\int_0^1 f(x,u_m)v \to \int_0^1 f(x,u)v,\ m\to\infty$. Mat khac $\int_0^1 u_m v" \to \int_0^1 uv"$, $m\to\infty$ vi u_m hoi tu yeu den u trong H_0^1. Cuoi cung tu (5) ta duoc

(6) $\int_0^1 u(x)v"dx + \int_0^1 f(x,u(x))v(x)dx = 0\ \forall v\in H_0^1\cap H^2$

do tinh tru mat cua V_j trong $H_0^1\cap H^2$.

Trong (6) lay $v=\varphi\in D(\Omega)$ dan den $u\in H^2$ do cau hoi 1) va tich phan $\int_0^1 u\varphi" = \int_0^1 u"\varphi$ co nghia. Cuoi cung ham u thoa dung phuong trinh vi phan

(7) $u"(x) + f(x,u(x)) = 0$, theo nghia phan bo.

Tinh lien tuc cua f cong them $u\in H_0^1$ dan den $u\in C^2(\overline{\Omega})$.

4. Lap cong thuc bien phan cua (7) ta co

(8) $\int_0^1 u'v'dx = \int_0^1 f(x,u)vdx\ \forall v\in H_0^1$

Lay $v=-u^-\in H_0^1$ trong (8). Cho nen $\int_0^1 \Big(\dfrac{du^-}{dx}\Big)^2dx = -\int_0^1 f(x,u)u^-dx \le 0$ do gia thiet ve f. Ma $u^-\in H_0^1$ dan den $\int_0^1 (u^-)^2dx \le 0$ (do bat dang thuc cua Poincare) tuc la $u^-=0$. Vay $u=u^+\ge 0$.

5. a) Cho $u_1,\ u_2$ hai loi giai cua bai toan (P). Ta co

$$\int_0^1 (u_1'-u_2')^2dx = -\int_0^1 (u_1-u_2)(u_1" - u_2")dx = \int_0^1 (u_1-u_2)(f(x,u_1) - f(x,u_2)dx \le 0$$

dan den $\int_0^1 (u_1'-u_2')^2dx = 0$ tuc la $u_1-u_2 = C$ hang so. u_1 va u_2 thuoc H_0^1 keo theo $u_1=u_2$ chung minh tinh duy nhat cua (P).

b) Ca day $\{u_m\}$ hoi tu yeu trong H_0^1 den u. Neu cai do sai thi co ton tai mot day con $\{u_\mu\}\subset\{u_m\}$ sao cho u_μ khong tien toi u trong H_0^1 yeu tuc la

(9) $\exists v_0\in H_0^1\ |< u_\mu, v_0 >_{H_0^1}$ khong tien toi $< u, v_0 >_{H_0^1}$ khi μ di den vo cuc.

Ca bien luan truoc van con gia tri voi $m = \mu$. Do do co mot day con $\{u_\nu\} \subset \{u_\mu\}$ sao cho $u_\nu \to u$ trong H_0^1 yeu vi duy nhat giai. Vay thi mau thuan voi (9) neu ta lay v bang v_0 trong hoi tu yeu.

c) Ta chung minh bay gio hoi tu manh cua u_m den u trong H^2. Cong thuc (3) co the viet lai duoi hinh thuc

(10) $< J'(u_m), \varphi >_{L2}= \int_0^1 (-u_m"(x) - f(x, u_m(x))\varphi(x)dx \ \forall \varphi \in V_m.$

Vi the $u"_m(x) = -P_m[f(x, u_m)]$, trong do P_m la phep chieu truc giao cua $L^2(\Omega)$ tren V_m. Ta co $||u"_m||_0 \leq ||P_m||_0||f(x, u_m)||_0 \leq M \Longrightarrow \{u_m\}$ nam trong mot tap hop bi chan cua $H_0^1 \cap H^2$. Vay thi $u_m \to u$ trong $C^0(\overline{\Omega}) \cap H_0^1$ do tinh phep nhung compac cua H^2 trong H_0^1 va H_0^1 trong $C^0(\overline{\Omega})$. Chung minh cai hoi tu manh cua u_m den u trong H^2 phai danh gia $||u"_m - u"||_0$. Ta co

(11) $||u"_m - u"||_0 \leq || - P_m[f(x, u_m)] + f(u||_0 \leq$
$||P_m[f(x, u_m)] - P_m[f(x, u)]||_0 + ||P_m[f(x, u)] - f(x, u)||_0$

Ta co $||P_m[f(x, u_m)] - P_m[f(x, u)]||_0 \leq ||f(x, u_m) - f(x, u)||_0 \to 0$ khi $m \to \infty$ do dinh ly cua Lebesgue. Mat khac $||P_m[f(x, u)] - f(x, u)||_0 \to 0$ khi $m \to \infty$ vi V_m tru mat trong $L^2(\Omega)$. Vay thi do (11) $||u"_m - u"||_0 \to 0$ khi $m \to \infty$, chung minh hoi tu manh cua u_m den u trong H^2.

6.
- dim $W_h = N - 1$
- cong thuc bien phan cua bai toan xap xi:

(12) $\int_0^1 \frac{du_h^{(n)}}{dx} v_h' dx = \int_0^1 f(x, u_h^{(n-1)}) v_h dx, \ \forall v_h \in W_h$

- ma tran cung cua he thong (12)

$$A_h = \frac{1}{h} \begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & 2 & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$