On the twisted Alexander polynomial and the A-polynomial of 2-bridge knots

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Abstract. We show that the A-polynomial $A(L, M)$ of a 2-bridge knot $\mathfrak{b}(p, q)$ is irreducible if p is prime, and if $(p-1)/2$ is also prime and $q \neq 1$ then the L-degree of $A(L, M)$ is $(p-1)/2$. This shows that the AJ conjecture relating the A-polynomial and the colored Jones polynomial holds true for these knots, according to work of the second author. We also study relationships between the A-polynomial of a 2-bridge knot and a twisted Alexander polynomial associated with the adjoint representation of the fundamental group of the knot complement. We show that for twist knots the A-polynomial is a factor of the twisted Alexander polynomial.

1. Background and conventions

1.1. Representation variety. Let K be a knot in S^3 and $X = S^3 \setminus K$ be its complement. Let $\pi = \pi_1(X)$ be the fundamental group of the complement. Let $R(\pi) = \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$ be the set of representations of π to $\mathrm{SL}(2, \mathbb{C})$. This is a complex affine algebraic set, which is called the *representation variety*, although it might be a union of a finite number of (irreducible) algebraic varieties in the sense of algebraic geometry. The group $\mathrm{SL}(2, \mathbb{C})$ acts on $R(\pi)$ by conjugation. The algebro-geometric quotient of $R(\pi)$ under this action is called the *character variety* of π, denoted by $X(\pi)$. The character of a representation ρ is the map $\chi_\rho : \pi \to \mathbb{C}$ determined by $\chi_\rho(\gamma) = \mathrm{tr}\,\rho(\gamma)$, for $\gamma \in \pi$. There is a bijection between $X(\pi)$ and the set of characters of representations of π.

1.2. The A-polynomial. Let $B = (\mu, \lambda)$ be a pair of meridian-longitude of the boundary torus of X. Let R_U be the subset of $R(\pi)$ containing all representations ρ such that $\rho(\mu)$ and $\rho(\lambda)$ are upper triangular matrices:

$$\rho(\mu) = \begin{pmatrix} M & * \\ 0 & M^{-1} \end{pmatrix}, \quad \rho(\lambda) = \begin{pmatrix} L & * \\ 0 & L^{-1} \end{pmatrix} \tag{1.1}$$

(any representation can be conjugated to have this form). Then R_U is an algebraic set, because we only add the requirement that the lower left entries of $\rho(\mu)$ and $\rho(\lambda)$

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are zeros. Define the projection map $\xi : R_U \to \mathbb{C}^2$ by $\xi(\rho) = (L, M)$. Consider the Zariski closure $\overline{\xi(R_U)}$ of the projection $\xi(R_U) \subset \mathbb{C}^2$. It is known that $\overline{\xi(R_U)}$ is an algebraic set whose components have dimensions zero or one. If a component has dimension one then it is a curve defined by a single polynomial in L and M. The product of these polynomials, divided by $L - 1$, is called the *A-polynomial* of K [1]. The reason for dividing by $L - 1$ is as follows. If ρ is an abelian representation then it factors through $H_1(X) = \langle\mu\rangle$, so $\rho(\lambda)$ is the identity matrix, therefore the component of $\overline{\xi(R(_U))}$ corresponding to abelian representations is defined by a single equation $L = 1$. Thus in the construction of the A-polynomial one can restrict to nonabelian representations.

It is known that a multiple constant can be chosen so that the A-polynomial is an integer polynomial. We assume that the A-polynomial has no repeated factors; and that it has no integer factors, i.e. its coefficients are coprime. If instead of the basis $B = (\mu, \lambda)$ we choose the other basis (μ^{-1}, λ^{-1}) then the pair (L, M) is replaced by the pair (L^{-1}, M^{-1}) as can be seen from (1.1), and it is known that $A_K(L^{-1}, M^{-1}) = \pm L^m M^n A_K(L, M)$. Thus $A_K(L, M)$ is an integer polynomial defined up to a factor $\pm L^m M^n$.

With finitely many exceptions, corresponding to a pair (L, M) satisfying $A(L, M) = 0$ there is a nonabelian representation $\rho \in R(\pi)$ for which (1.1) holds.

For more on the A-polynomial we refer to [CCG+94], [CL96] and [CL98].

1.3. 2-bridge knots. Let $p = 2n + 1$, $n \geq 1$, and $0 < q < p$, q is odd, $\gcd(p, q) = 1$. The fundamental group of the complement X of the 2-bridge knot $\mathfrak{b}(p, q)$ has a presentation $\pi = \pi_1(X) = \langle a, b / wa = bw\rangle$, where both a and b are meridians. The word w has the form $a^{\epsilon_1} b^{\epsilon_2} a^{\epsilon_3} b^{\epsilon_4} \cdots a^{\epsilon_{2n-1}} b^{\epsilon_{2n}}$, where $\epsilon_i = (-1)^{\lfloor iq/p \rfloor}$. In particular, if we read w from right to left and interchange a and b then we get w again. For example, $\mathfrak{b}(2n + 1, 1)$ is the torus knot $T(2, 2n + 1)$, and in this case $w = (ab)^n$.

We adopt the convention that if $\rho \in R(\pi)$ and x is a word then we write $\operatorname{tr} x$ for $\operatorname{tr} \rho(x)$. Let $x = \operatorname{tr} a$ and $y = \operatorname{tr} ab$. Thang Le [Le93] showed that the character variety $X^{nab}(\pi)$ of nonabelian representations of π is determined by the polynomial $\mathbf{\Phi}_{(p,q)}(x, y) = \operatorname{tr} w - \operatorname{tr} w' + \cdots + (-1)^{n-1} \operatorname{tr} w^{(n-1)} + (-1)^n$, here if x is a word then x' denotes the word obtained from x by deleting the two letters at the two ends.

For more on 2-bridge knots see [BZ03], and for representations of 2-bridge knot groups we refer to [Ril84] and [Le93].

1.4. Nonabelian and irreducible representations. A representation ρ is said to be *reducible* if the action (i.e. the linear map) it induces on $\mathbb{C}^2$ fix a one dimensional

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[1] The letter A stands for *affine* – according to Garoufalidis.

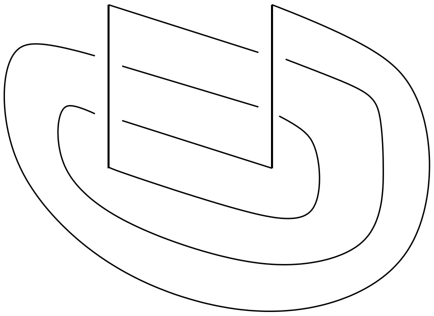

FIGURE 1. The trefoil as the 2-bridge knot $\mathfrak{b}(3,1)$.

subspace of $\mathbb{C}^2$. This is equivalent to saying that ρ can be conjugated to be a representation by upper triangular matrices (one can take an eigenvector of the linear map as a new basis vector for $\mathbb{C}^2$). Otherwise ρ is said to be *irreducible*.

An elementary argument (as suggested above) would show that if ρ is irreducible then it is nonabelian. For 2-bridge knots we have a stronger result ([Le93]): Except finitely many cases, a nonabelian representation is irreducible. The Zariski closure $\overline{X^{irr}(\pi)}$ of the set of characters of irreducible representations is exactly the character variety $X^{nab}(\pi)$ of nonabelian representations. Therefore in some arguments we can consider irreducible representations instead of nonabelian representations.

1.5. The A-polynomial of 2-bridge knots. Suppose that ρ is an irreducible representation. After conjugations if necessary we may assume that

$$\rho(a) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} M & 0 \\ -z & M^{-1} \end{pmatrix}. \tag{1.2}$$

We have $x = \operatorname{tr} a = M + M^{-1}$ and $z = x^2 - 2 - y$ where $y = \operatorname{tr} ab$. Let $\lambda = w\overleftarrow{w}b^{-2e}$, where $\overleftarrow{w}$ is the word obtained from w by writing the letters in w in reversed order (i.e. by interchanging a and b), and e is the sum of the exponents of the letters in w. Then λ represents the longitude of the boundary torus of the knot complement, and we define $\mathscr{L}(M, y)$ to be the upper left entry of the matrix $\rho(\lambda)$. Then up to a factor of the form an integral power of M, $\mathscr{L}(M, y)$ is a polynomial. Because $x = M + M^{-1}$ we can consider $\mathbf{\Phi}$ as a function in M and y, up to a factor of the form M to an integral power it is a polynomial. The A-polynomial $A(L, M)$ can be computed by deleting repeated factors from the resultant $\operatorname{Res}(\mathbf{\Phi}(M, y), \mathscr{L}(M, y) - L)$, where the resultant is computed with respect to y.

The description above can be implemented for computer calculations.

Example 1.1. The A-polynomial of $\mathfrak{b}(3,1)$ (the trefoil) is $LM^6 + 1$, and that of $\mathfrak{b}(5,3)$ (the figure-8 knot) is $-LM^8 + LM^6 + L^2M^4 + 2LM^4 + M^4 + LM^2 - L$.

For further details on the A-polynomial of 2-bridge knots we refer to [CCG+94] and [HS04].

1.6. The adjoint representation. The Lie algebra $\mathrm{sl}_2(\mathbb{C})$ of $\mathrm{SL}(2,\mathbb{C})$ consists of 2×2 matrices with zero traces. Consider the adjoint representation of $\mathrm{SL}(2,\mathbb{C})$, $Ad : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Aut}(\mathrm{sl}_2(\mathbb{C}))$. For $A \in \mathrm{SL}(2,\mathbb{C})$ and $x \in \mathrm{sl}_2(\mathbb{C})$ we have $Ad_A(x) = AxA^{-1}$. Since $\mathrm{sl}_2(\mathbb{C})$ can be identified with $\mathbb{C}^3$, Ad_A is a linear map on $\mathbb{C}^3$ and it turns out that it belongs to $\mathrm{SO}(3,\mathbb{C})$. If $\rho \in R(\pi)$ then the composition $Ad \circ \rho$ is a representation of π to $\mathrm{SO}(3,\mathbb{C})$.

2. Irreducibility of the A-polynomial of 2-bridge knots

2.1. Introduction. In his recent study on the AJ conjecture which relates the A-polynomial and the colored Jones polynomial of a knot, Thang Le [Le04] proved that for a 2-bridge knot $\mathfrak{b}(p,q)$ the AJ conjecture holds true if the A-polynomial is irreducible and has L-degree $(p-1)/2$. In this chapter we will provide a proof for the result (Theorem 2.5 below) that the above condition is satisfied if both p and $(p-1)/2$ are prime and $q \neq 1$.

In a related result, recently Hoste and Shanahan [HS04] using trace field theory have proved that the A-polynomial of the twist knot K_n, which is the 2-bridge knot $\mathfrak{b}(4n+1, 2n+1)$, is irreducible. From their recursive formula it can be checked easily that the L-degree is exactly $2n$.

2.2. Proofs. Let $\mathbf{\Phi}_n(x,y) = \mathbf{\Phi}_{(p,1)}(x,y)$, where $p = 2n+1$. It has been shown in [Le93, Proposition 4.3.1] (also see below) that $\mathbf{\Phi}_n(x,y)$ does not depend on x.

Proposition 2.1. *$\mathbf{\Phi}_n(y)$ is irreducible if and only if $2n+1$ is prime.*

Proof. It is immediate from [Le93, Proposition 4.3.1] that $\mathbf{\Phi}_n(2y) = (T_n(y) + T_{n+1}(y))/(y+1)$, where T_n is the nth Chebyshev polynomial (of the first kind). Let $\tilde{\mathbf{\Phi}}_n(y) = \mathbf{\Phi}_n(2y)$. It is well-known that by letting $\theta = \cos y$, we can write $T_n(y) = \cos(n\theta)$, and so $\tilde{\mathbf{\Phi}}_n(\theta) = \cos((2n+1)\frac{\theta}{2}) / \cos(\frac{\theta}{2})$. It also follows that $\tilde{\mathbf{\Phi}}_n(y)$ is an integer polynomial of degree n with exactly n roots given by $y = \cos(\frac{2k+1}{2n+1}\pi)$, $0 \le k \le n-1$. Fix $\theta = \pi/p$. Noting that $\tilde{\mathbf{\Phi}}_n$ has no integer factor since $\tilde{\mathbf{\Phi}}_n(0) = \pm 1$ we see that $\tilde{\mathbf{\Phi}}_n$ is irreducible, and so is $\mathbf{\Phi}_n$, if and only if the extension field degree $[\mathbb{Q}(\cos\theta) : \mathbb{Q}]$ is exactly the degree of $\tilde{\mathbf{\Phi}}_n$.

Noticing that $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, we want to study the extension field $\mathbb{Q}(e^{i\theta})$. It is well-known (see, e.g. [Lan93, p. 276]) that the irreducible polynomial of $e^{i\theta}$ is the cyclotomic polynomial

$$C_{2p}(y) = \prod_{1\le d\le 2p,\ (d,2p)=1} (x - e^{d\pi i/p}).$$

This is an integer polynomial whose degree is $\varphi(2p) = \varphi(p)$, here φ is the Euler totient function. Thus the degree of the extension field is $[\mathbb{Q}(e^{i\theta}) : \mathbb{Q}] = \varphi(p)$. From the identity $(x - e^{\theta i})(x - e^{-\theta i}) = x^2 - 2(\cos\theta)x + 1$, we see that $[\mathbb{Q}(e^{i\theta}) : \mathbb{Q}(\cos\theta)] = 2$, thus $[\mathbb{Q}(\cos\theta) : \mathbb{Q}] = \varphi(p)/2$. Therefore $\mathbf{\Phi}_n$ is irreducible if and only if $\varphi(p) = p - 1$, which happens if and only if p is prime. □

Proposition 2.2. *We have* $\mathbf{\Phi}_{(p,q)}(0, y) = \mathbf{\Phi}_{(p,1)}(y)$. *Hence if* $\mathbf{\Phi}_{(p,1)}(y)$ *is irreducible then* $\mathbf{\Phi}_{(p,q)}(x, y)$ *is also irreducible.*

Proof. Recall from section 1.5 that we can write $\rho(a) = \left(\begin{smallmatrix} M & 1 \\ 0 & M^{-1}\end{smallmatrix}\right)$ and $\rho(b) = \left(\begin{smallmatrix} M & 0 \\ -z & M^{-1}\end{smallmatrix}\right)$, where $M + M^{-1} = x$ and $z = x^2 - 2 - y$. If $x = \operatorname{tr} a = \operatorname{tr} b = M + M^{-1} = 0$ then it is immediate that $\rho(a^{-1}) = -\rho(a)$ and $\rho(b^{-1}) = -\rho(b)$ (this can also be seen from the Cayley-Hamilton Theorem: the characteristic polynomial of $\rho(a)$ is $t^2 - (\operatorname{tr} a)t + 1$). Recall that $\mathbf{\Phi}_{(p,q)}(x, y) = \operatorname{tr} w - \operatorname{tr} w' + \cdots + (-1)^{n-1} \operatorname{tr} w^{(n-1)} + (-1)^n$. Because the word w is palindromic, so is each word $w^{(i)}$, $0 \le i \le n - 1$, and hence in $w^{(i)}$ we have a^{-1} and b^{-1} appear in pairs. That means $\rho(w^{(i)})$ does not change if we replace a^{-1} by a and b^{-1} by b. Thus $\rho(w^{(i)}) = \rho((ab)^{n-i})$. Recalling that for a torus knot $\mathfrak{b}(p, 1)$ we have $w = (ab)^n$, the result follows. □

Because $x = M + M^{-1}$ we can consider $\mathbf{\Phi}$ as a function in M and y, and it is a polynomial up to a factor of the form M to an integral power, which is omitted.

Proposition 2.3. *If* $\mathbf{\Phi}(M, y)$ *is irreducible then* $A(L, M)$ *is irreducible.*

Proof. Recall from Section 1.5 that the A-polynomial $A(L, M)$ of a 2-bridge knot can be computed by deleting repeated factors from $\operatorname{Res}(\mathbf{\Phi}(M, y), \mathscr{L}(M, y) - L)$, where $\mathscr{L}(M, y)$ is a polynomial and the resultant is computed with respect to y.

We have $A(L, M) = 0$ if and only if there is y such that $\mathbf{\Phi}(M, y) = 0$ and $\mathscr{L}(M, y) = L$. Writing $Z(f)$ for the zero set of a polynomial f, we see that for each $(M, L) \in Z(A(L, M))$ there is $(M, y) \in Z(\mathbf{\Phi}(M, y))$ such that $(M, \mathscr{L}(M, y)) = (M, L)$.

In what follows we use some simple notions in algebraic geometry, which can be found for example in [Har77]. Consider the map $pr : \mathbb{C}^2 \to \mathbb{C}^2$ given by $pr(u, v) = (u, \mathscr{L}(u, v))$. This map is continuous under the Zariski topology. It projects $Z(\mathbf{\Phi}(M, y))$ onto $Z(A(L, M))$.

Note that f is an irreducible polynomial if and only if $Z(f)$ is an irreducible algebraic set. Now suppose that the A-polynomial is reducible, hence $Z(A(L, M))$ is a union of two nonempty closed subsets B and C. Then $pr^{-1}(B) \cap Z(\mathbf{\Phi})$ and

$pr^{-1}(C) \cap Z(\mathbf{\Phi})$ are two nonempty closed sets whose union is $Z(\mathbf{\Phi})$. This implies that $\mathbf{\Phi}(M, y)$ is reducible, a contradiction. □

Proposition 2.4. *If the L-degree of $A(L, M)$ is 1 then $q = 1$, and so $\mathfrak{b}(p, q)$ is the torus knot $T(2, p)$.*

The idea for the following proof was communicated to us by Nathan Dunfield. We also thank Xingru Zhang for a discussion on this topic.

Proof. We need the concept of Newton polygons of A-polynomials. The Newton polygon of $A(L, M)$ is the convex hull of the set of points (i, j) on the real LM-plane such that the coefficient a_{ij} of the term $a_{ij}L^iM^j$ of $A(L, M)$ is nonzero. The slopes of the sides of the Newton polygon are boundary slopes of incompressible surfaces in the knot complement ([CCG$^+$94]).

For example the following figure shows the Newton polygon of the torus knot $\mathfrak{b}(3, 1) = T(2, 3)$ (the trefoil) whose A-polynomial is $LM^6 + 1$, and that of $\mathfrak{b}(5, 3)$ (the figure-8 knot) whose A-polynomial is $-LM^8 + LM^6 + L^2M^4 + 2LM^4 + M^4 + LM^2 - L$.

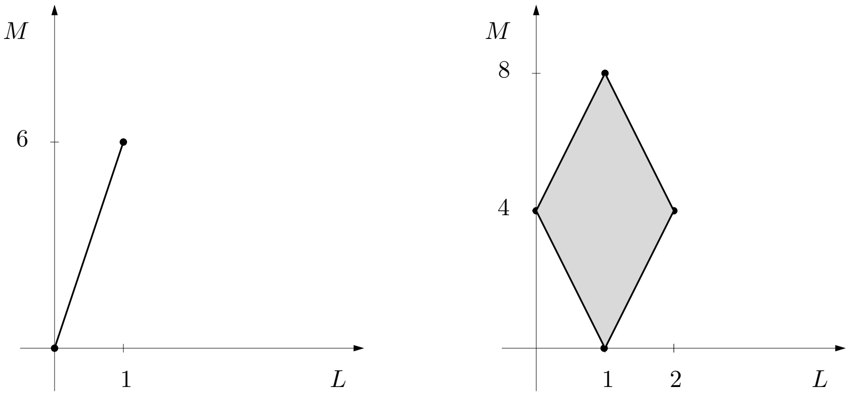

Figure 2. Newton polygons of the A-polynomials of $\mathfrak{b}(3, 1)$ and $\mathfrak{b}(5, 3)$.

Suppose that the L-degree of $A(L, M)$ is 1. This means that the Newton polygon either has ∞ as a slope, or has only one edge. The Hatcher-Thurston classification of incompressible surfaces in 2-bridge knot complements [HT85, Proposition 2] shows that actually ∞ cannot be a slope, in fact all boundary slopes are integers.

Thus the Newton polygon has only one edge. For a hyperbolic knot the Newton polygon has at least two distinct sides. Thus the knot is non-hyperbolic.

Since 2-bridge knots are alternating ([BZ03]) a theorem of Menasco [Men84] says that the knot can only be a torus knot. Since the bridge number of a torus knot $T(p, q)$ is at least $\min\{p, q\}$, the torus knot must be $T(2, p) = \mathfrak{b}(p, 1)$.

Note that for a torus knot $T(2, p)$ indeed $A(L, M) = LM^{2p} + 1$ ([HS04, Zha04]) having L-degree 1. $\square$

Theorem 2.5. *If p is prime then the A-polynomial of $\mathfrak{b}(p, q)$ is irreducible. Furthermore if $(p-1)/2$ is also prime and $q \neq 1$ then the L-degree of $A(L, M)$ is $(p-1)/2$.*

Proof. The first part follows from Propositions 2.1, 2.2 and 2.3. We prove the second part.

First we claim that the y-degree of $\mathbf{\Phi}_{(p,q)}(M, y)$ is $n = (p-1)/2$. Indeed, look at $\mathbf{\Phi}_{(p,q)}(M, y) = \operatorname{tr} w - \operatorname{tr} w' + \cdots + (-1)^{n-1} \operatorname{tr} w^{(n-1)} + (-1)^n$. Because the letter b appears n times in the word w, the entries of the matrix $\rho(w)$ have z-degrees, hence y-degrees, at most n. So the y-degree of $\mathbf{\Phi}_{(p,q)}(M, y)$ is at most n. On the other hand Proposition 2.2 and the proof of Proposition 2.1 show that the y-degree is at least n, so the claim follows.

From the determinant description of resultant ([Lan93, p. 200]) it is clear that $\operatorname{Res}(\mathbf{\Phi}(M, y), \mathscr{L}(M, y) - L)$ has degree n in L. Since $A(L, M)$ is irreducible we have a positive integer k such that $A^k(M, L) = \operatorname{Res}(\mathbf{\Phi}(M, y), \mathscr{L}(M, y) - L)$. Thus the L-degree ℓ of $A(L, M)$ must be a factor of n. If n is prime then ℓ can only be 1 or n. If $\ell = 1$ then the knot is a torus knot and $q = 1$ according to Proposition 2.4. $\square$

3. Twisted Alexander polynomial and the A-polynomial of 2-bridge knots

Definition 3.1. Let $\pi = \langle a, b / r = waw^{-1}b^{-1} = 1 \rangle$. Let ρ be the representation of the free group $\langle a, b \rangle$ defined by the formula

$$\rho(a) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} M & 0 \\ -z & M^{-1} \end{pmatrix}. \tag{3.1}$$

Extend the map $Ad \circ \rho$ linearly, and consider M and z as formal variables. The twisted Alexander polynomial $\Delta_K^{Ad}(M, z)$ associated to π is defined by

$$\Delta_K^{Ad}(M, z) = \gcd\{\det(Ad \circ \rho(\partial r/\partial a)), \det(Ad \circ \rho(\partial r/\partial b))\} \in \mathbb{C}[M^{\pm 1}, z^{\pm 1}].$$

It is a polynomial in M and z up to a factor $\pm M^m z^n$.

For each pair (L_0, M_0) such that $A_K(L_0, M_0) = 0$ there is a finite number of numbers $z_i \in \mathbb{C}$ such that both polynomial equations $\mathbf{\Phi}(M_0, z_i) = 0$ and $\mathscr{L}(M_0, z_i) = L_0$ are satisfied.

Proposition 3.2. *Except for finitely many pairs (L_0, M_0), if $A_K(L_0, M_0) = 0$ then $\Delta_K^{Ad}(M_0, z_i) = 0$.*

Proof. Except a finite number of pairs (L_0, M_0), if $A_K(L_0, M_0) = 0$ then there is an irreducible representation $\rho \in R(\pi)$ for which $\rho(\mu) = \left(\begin{smallmatrix} M_0 & * \\ 0 & M_0^{-1} \end{smallmatrix}\right)$, $\rho(\lambda) = \left(\begin{smallmatrix} L_0 & * \\ 0 & L_0^{-1} \end{smallmatrix}\right)$, and

$$\rho(a) = \begin{pmatrix} M_0 & 1 \\ 0 & M_0^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} M_0 & 0 \\ -z_i & M_0^{-1} \end{pmatrix}. \tag{3.2}$$

Following a standard argument, the knot complement X is simple homotopic to a 2-dimensional cell complex with one 0-cell, two 1-cells and one 2-cell. Letting $\widetilde{X}$ be the universal cover, we can consider the cochain complex of complex vector spaces:

$$0 \leftarrow \mathbb{C}^3 \otimes_{\mathbb{Z}[\pi], Ad\circ\rho} C^2(\widetilde{X}) \xleftarrow{\partial_2} \mathbb{C}^3 \otimes_{\mathbb{Z}[\pi], Ad\circ\rho} C^1(\widetilde{X}) \xleftarrow{\partial_1} \mathbb{C}^3 \otimes_{\mathbb{Z}[\pi], Ad\circ\rho} C^0(\widetilde{X}) \leftarrow 0.$$

Here ∂_2 is represented by the 3×6-matrix $\left(Ad\circ\rho(\partial r/\partial a) \;\; Ad\circ\rho(\partial r/\partial b) \right)$ and ∂_1 is represented by the 6×3-matrix $\left(\begin{smallmatrix} Ad\circ\rho(a-1) \\ Ad\circ\rho(b-1) \end{smallmatrix}\right)$. A direct computation shows that $Ad \circ \rho(b-1)$ is non-singular. Thus $\operatorname{rank}(\operatorname{Im}\partial_1) = 3$. The first cohomology group with local coefficients of X is $H^1_{Ad\circ\rho}(X) = \ker\partial_2 / \operatorname{Im}\partial_1$.

At this point we use a theorem of Weil [Wei64] (see [Por97, p. 69], [BZ00]). The theorem asserts that if ρ is an irreducible representation then the Zariski tangent $T^{Zar}_{\chi_\rho}(X(\pi))$ of the character variety $X(\pi)$ at the point χ_ρ is isomorphic as complex vector space to a subspace of the first cohomology group $H^1_{Ad\circ\rho}(X)$. For the Zariski tangent space at a point P of an algebraic variety Y we always have $\operatorname{rank} T^{Zar}_P(Y) \geq \operatorname{rank}(Y)$. In this case because the point χ_ρ arises from a point on the curve defined by $A(L, M)$, the dimension of the irreducible component of $X(\pi)$ containing χ_ρ is at least one (we can also evoke a theorem of Thurston to this effect, see e.g. [CS83, Proposition 3.2.1]). Thus $\operatorname{rank} T^{Zar}_{\chi_\rho}(X(\pi)) \geq 1$, hence $\operatorname{rank} H^1_{Ad\circ\rho}(X) \geq 1$.

Since $\operatorname{rank}(\ker\partial_2 / \operatorname{Im}\partial_1) \geq 1$ and $\operatorname{rank}(\operatorname{Im}\partial_1) = 3$ it follows that $\operatorname{rank}(\ker\partial_2) \geq 4$, hence $\operatorname{rank}(\operatorname{Im}\partial_2) \leq 2$. This means that both 3×3-matrices $Ad \circ \rho(\partial r/\partial a)$ and $Ad \circ \rho(\partial r/\partial b)$ have ranks less than three and thus are singular. Hence $\det(Ad \circ \rho(\partial r/\partial a)) = \det(Ad \circ \rho(\partial r/\partial b)) = 0$. This means $\Delta^{Ad}_K(M, z)$ vanishes when it is evaluated at (M_0, z_i). □

In the special case of a twist knot K_n, which is the 2-bridge knot $\mathfrak{b}(4n+1, 2n+1)$, it is shown in [HS04, p. 203] (note that $K_n = J(2, -2n)$ in their notation) that the correspondence $z_i \mapsto L_0$ is one-to-one. Specifically z can be expressed in terms of L and M as

$$z = \frac{(1-L)(1-M^2)}{L+M^2}. \tag{3.3}$$

Using this change of variable we can write the twisted Alexander polynomial $\Delta^{Ad}_K(M, z)$ as a polynomial $\Delta^{Ad}_K(L, M)$.

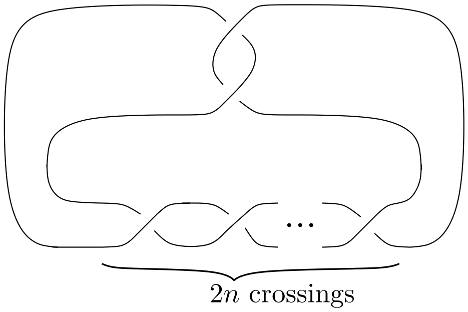

FIGURE 3. The twist knot $K_n, n > 0$.

THEOREM 3.3. *If K is twist knot then the polynomial $A_K(L, M)$ is a factor of the polynomial $\Delta_K^{Ad}(L, M)$.*

Proof. For a twist knot Proposition 3.2 says that the zero set $Z(A)$ of the A-polynomial $A(L, M)$ minus a set I consists of finitely many points is contained in the zero set $Z(\Delta^{Ad})$ of the twisted Alexander polynomial $\Delta^{Ad}(L, M)$. The Zariski closure of $Z(A) \setminus I$ is exactly $Z(A)$. Thus we have $Z(A) \subset Z(\Delta^{Ad})$ and so $A(L, M)$ is a factor of $\Delta^{Ad}(L, M)$. □

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