REIDEMEISTER TORSION AND CIRCLE-VALUED MORSE THEORY

NOTES FOR A GRADUATE STUDENT SEMINAR BY VU QUANG HUYNH

ABSTRACT. Morse theory and the theory of Reidemeister torsion are two classical subjects of topology, both originated in the 1930's and have been developed ever since. In recent works M. Hutchings and J. Y. Lee have found a connection between the two subjects. This note aims to present some background needed for describing their result.

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1. Reidemeister torsion

1.1. Torsion of a chain complex.

References for this section are Turaev[11] and Milnor[7].

Let $\mathbb{F}$ be a field, V be a k-dimensional vector space over $\mathbb{F}$. Suppose that $b = (b_1, b_2, ..., b_k)$ and $c = (c_1, c_2, ..., c_k)$ are two ordered bases of V. Then there is a non-singular $k \times k$ matrix (a_{ij}) such that

$$c_i = \sum_{j=1}^{k} a_{ij} b_j.$$

We write

$$[c/b] = \det(a_{ij}) \in \mathbb{F}^* = \mathbb{F} \setminus \{0\}.$$

By linear algebra, we have $[b/b] = 1$, and if d is another basis then $[d/b] = [d/c].[c/b]$.

We call two bases b and c *equivalent* if $[b/c] = 1$. The above properties show that this is indeed an equivalence relation. We will identify a basis with its equivalence class. Also we say b and c have the same *orientation* if $[b/c] > 0$.

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Let $0 \to C \overset{\alpha}{\hookrightarrow} D \overset{\beta}{\twoheadrightarrow} E \to 0$ be a short exact sequence of vector spaces. Then $\dim D = \dim C + \dim E$. Let $c = (c_1, ..., c_k)$ be a basis for C, $e = (e_1, ..., e_l)$ be a basis for E. Since β is surjective, we can lift e_i to some vector $\widetilde{e_i}$ in D. Then using linear algebra we can prove easily that $ce = (c_1, ..., c_k, \widetilde{e_1}, ..., \widetilde{e_l})$ is a basis for D and its equivalent class does not depend on the choice of $\widetilde{e_i}$, only on the equivalence classes of c and e.

Let $C_0, C_1, ..., C_m$ be finite dimensional vector spaces over $\mathbb{F}$ and $\partial_i : C_{i+1} \to C_i$ linear homomorphisms. The chain complex

$$(C, \partial) = (0 \to C_m \overset{\partial_{m-1}}{\to} C_{m-1} \overset{\partial_{m-2}}{\to} ... \overset{\partial_1}{\to} C_1 \overset{\partial_0}{\to} C_0 \to 0)$$

is called *acyclic* if the chain is exact. In that case $H_*(C) = 0$. (C, ∂) is called *based* if for each C_i a basis is chosen.

Assume that (C, ∂) is acyclic and based. Let $B_i = \mathrm{Im}(\partial_i : C_{i+1} \to C_i) \subset C_i$. Choose a basis b_i for B_i. We have $C_i/\mathrm{Ker}(\partial_{i-1}) = \mathrm{Im}(\partial_{i-1}) = B_{i-1}$. Since $\mathrm{Ker}(\partial_{i-1}) = \mathrm{Im}(\partial_i) = B_i$, we get $C_i/B_i = B_{i-1}$. In other words we have the following short exact sequence:

$$0 \to B_i \hookrightarrow C_i \twoheadrightarrow B_{i-1} \to 0.$$

By the previous argument $b_i b_{i-1}$ is a basis of C_i. But C_i already has a basis c_i.

Definition 1.1. *The torsion of the acyclic and based chain complex C is defined to be*

$$\tau(C) = \prod_{i=0}^{m} [b_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{F}^*.$$

The torsion $\tau(C)$ does not depend on the choice of the b_i's. If we use a basis c_i' for C_i instead of c_i then the torsion is multiplied with

$$[c_i/c_i']^{(-1)^{i+1}}. \tag{1}$$

In particular if we change the orientation of the basis of C_i then the torsion changes sign.

Change of rings. Now let R be a ring and (C, ∂) be a based chain complex over R. Suppose $\mathbb{F}$ is a field and $\varphi : R \longrightarrow \mathbb{F}$ is a ring homomorphism.

Then by using φ we can consider $\mathbb{F}$ as an R module, namely we define for $a \in R$ and $b \in \mathbb{F}$ the product $a \cdot b := \varphi(a)b$. Thus we can form the tensor $C_i \otimes_\varphi \mathbb{F}$ of free modules over $\mathbb{F}$. Moreover $C_i \otimes_\varphi \mathbb{F}$ is a vector space over $\mathbb{F}$: if $\{e_1, e_2, \ldots, e_n\}$ is a basis for C_i then $\{e_1 \otimes 1, e_2 \otimes 1, \ldots, e_n \otimes 1\}$ is a basis for $C_i \otimes_\varphi \mathbb{F}$, here 1 is the unit of $\mathbb{F}$. The boundary map $\partial_i : C_{i+1} \longrightarrow C_i$ of the chain complex C induces a boundary map for $C_i \otimes_\varphi \mathbb{F}$. Thus $(C_* \otimes_\varphi \mathbb{F})$ becomes a based chain complex of vector spaces over $\mathbb{F}$. If $C_i \otimes_\varphi \mathbb{F}$ is acyclic then we can define its torsion $\tau(C_* \otimes_\varphi \mathbb{F}) \in \mathbb{F}^*$.

1.2. **Torsion of a CW-complex.** Let M be a finite connected CW-complex with cells $\{e_1, e_2, \ldots, e_n\}$. The *universal cover* $\widetilde{M}$ of M has a canonical CW-complex structure, obtained by lifting the cells of M. The group of covering transformations of $\widetilde{M}$ is $\pi_1 := \pi_1(M)$ and it acts freely and transitively on $\widetilde{M}$. Moreover, if $\tilde{e}$ and $\tilde{e}'$ are two liftings of the same cell e, then there is a unique element of π_1 that brings $\tilde{e}$ to $\tilde{e}'$. Thus this action of π_1 on $\widetilde{M}$ can be extended canonically to an action of π_1 on the cellular chain groups $C_i(\widetilde{M})$. Then extend this π_1-action linearly to a $\mathbb{Z}[\pi_1]$-action on $C_i(\widetilde{M})$, and $C_i(\widetilde{M})$ becomes a $\mathbb{Z}[\pi_1]$-module. If $\{e_i^k\}$ is the set of k-cells of M and $\tilde{e}_i^k$ is a lift of e_i^k then $\{\tilde{e}_i^k\}$ is a basis of the $\mathbb{Z}[\pi_1]$-module $C_i(\widetilde{M})$.

Suppose that $\mathbb{F}$ is a field and $\mathbb{Z}[\pi_1] \xrightarrow{\varphi} \mathbb{F}$ is a ring homomorphism. Then by the change of rings construction in the previous section, $C_*(\widetilde{M}) \otimes_\varphi \mathbb{F}$ becomes a chain complex over $\mathbb{F}$. If this chain complex is acyclic then we can define its torsion $\tau(C_*(\widetilde{M}) \otimes_\varphi \mathbb{F}) \in \mathbb{F}^*$.

We need to discuss how $\tau(C_*(\widetilde{M}) \otimes_\varphi \mathbb{F})$ depends on the choices involving $\widetilde{M}$ made above.

If we fix a choice of a set of lifting cells as a basis for the $\mathbb{Z}[\pi_1]$-module $C_i(\widetilde{M})$, but change the order of the cells in the basis then by formula (1) $\tau(C_*(\widetilde{M}) \otimes_\varphi \mathbb{F})$ is multiplied with ± 1. If we choose a different lifting cell for e_i^k-by an action $h \cdot \tilde{e}_i^k$ of a covering transformation $h \in \pi_1$ -then the torsion is multiplied with $\varphi(h)^{\pm 1}$.

Now we define the *torsion* $\tau^\varphi(M)$ *of the CW-complex* M to be the image of $\tau(C_*(\widetilde{M}) \otimes_\varphi \mathbb{F})$ under the quotient map $\mathbb{F} \longrightarrow \mathbb{F}/ \pm \varphi(\pi_1(M))$.

Remark. It is clear that the above construction can be used for any regular covering, not only for the universal cover.

1.3. **Turaev's refined torsion.** In the late 1980's V. Turaev constructed "refined torsions" in order to remove the ambiguities in the definition of Reidemeister torsion. Here we only describe the H_1-refined torsion [10].

Let M be a finite CW-complex whose Euler characteristic $\chi(M) = 0$. Denote by $A = \{e_1, \ldots, e_n\}$ the set of all cells of M. A *combinatorial Euler chain* on M is a 1-singular chain ξ such that its boundary has the form

$$\partial\xi = \sum_{e\in A}(-1)^{\mathrm{dim} e} x_e$$

where x_e is a point in e.

The fact that $\chi(M) = 0$ implies that the set of Euler chains is non-empty. For example, take a *spider*

$$\sum_{e\in A} Ox_e,$$

where O is a point in M, x_e is a point in e; Ox_e denotes a curve from O to x_e if dim$e$ is even and from x_e to O if dim$e$ is odd.

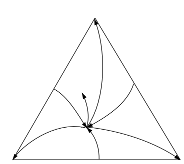

FIGURE 1. A spider.

Consider two Euler chains ξ and η with $\partial\xi = \sum_{e\in A}(-1)^{\mathrm{dim} e} x_e$ and $\partial\eta = \sum_{e\in A}(-1)^{\mathrm{dim} e} y_e$. Let $x_e y_e$ be a curve in e connecting x_e and y_e. The two

Euler chains are called *equivalent* if the 1-singular cycle

$$\xi - \eta + \sum_{e \in A} (-1)^{\mathrm{dim} e} x_e y_e$$

is homologous to 0.

One can check easily that this is indeed an equivalent relation and from now on we will identify an Euler chain with its equivalence class. Denote the quotient set by $\mathrm{Eul}(M)$. It can be shown that every Euler chain is equivalent to a spider???.

We have an action of $H_1 := H_1(M)$ on $\mathrm{Eul}(M)$ defined as $h[\xi] := [h + \xi]$. It can be checked easily that this action is free and transitive.

Another notion of interest is the fundamental family of cells. Let $\widetilde{M}$ be the *maximal abelian cover* of M, then $\widetilde{M}$ has an induced CW-complex structure from M. A *fundamental family of cells in* $\widetilde{M}$ is a set of the form $\{\tilde{e_1}, \ldots, \tilde{e_n}\}$ where each $\tilde{e_i}$ is a lifting cell of $e_i \in A$.

It is clear that H_1 acts freely and transitively on the set of fundamental families of cells (compare section 1.2 on torsion of CW-complex).

There is an H_1-isomorphism between the set of Euler chains and the set of fundametal families of cells. Namely, every Euler chain is equivalent to a spider, and every spider, when lifted, corresponds to a fundamental family of cells.

Now recall that in the definition of torsion of a CW-complex in section 1.2, we have the torsion $\tau^{\varphi}(M)$ belongs to $\mathbb{F}/\pm\varphi(H_1(M))$. The ambiguity H_1 shows up because we did not specify a particular fundamental family of cells in $\widetilde{M}$ when we constructed the torsion. Therefore we can get rid of the H_1 ambiguity by including a choice of the fundamental family of cells in $\widetilde{M}$, or equivalently, by including achoice of an Euler chain in M. Thus if $\xi \in \mathrm{Eul}(M)$ then the *Turaev's H_1-refined torsion* $\tau^{\varphi}(M, \xi)$ is defined and belongs to $\mathbb{F}^*/\pm 1$.

2. Morse theory

2.1. **The Morse complex.** Since the Novikov complex is a generalization of the Morse complex, we recall here the construction of the Morse complex for motivation. For details see, for instance [1]. For a historical survey of this subject, see Bott[2].

Let M be a smooth, closed, oriented, Riemannian manifold of dimension n. Let f be a smooth function on M.

A gradient flow line $\phi_t(x)$ of f on M is a solution of the initial differential equation corresponds to the negative gradient vector field of f :

$$\phi_t\prime(x) = -\nabla f(\phi_t(x)),$$

$$\phi_0(x) = x.$$

Note that these flow lines always point "downward", i.e. $f(\phi_t(x))$ is a decreasing function of t except at singular points of f.

Let x to be a critical point of f. The set of all points on the flow lines that start at x is called the *unstable manifold* (or descending manifold) of x, denoted by $W^u(x)$. So $W^u(x) = \{p \in M : \phi_t(p) \to \alpha \quad \text{as} \quad t \to -\infty\}$. Similarly, the *stable manifold* (or ascending manifold) $W^s(x) = \{x \in M : \phi_t(x) \to \alpha \quad \text{as} \quad t \to +\infty\}$ is the set of all points on the flow lines that end at x.

It is known that for f belongs to a dense set of smooth functions on M (which is called the generic set), then f is a Morse function (i.e. has non-degenerate critical points), all the stable and unstable manifolds intersect tranversally and they also intersect any regular level set $f^{-1}(a)$ transversally. In this section we will assume this condition, which is called the *Morse-Smale condition.*

Orientation of trajectories. Choose arbitrary orientations for all the unstable manifolds. Since $W^u(x)$ and $W^s(x)$ intersect tranversally and M is oriented, the orientation of $W^u(x)$ gives rise to an orientation of $W^s(x)$. The space of the trajectories joining two critical points x and y is $W^u(x) \cap W^s(y)$. Since $\dim W^u(x) = \text{ind}(x)$ and $\dim W^s(y) = n - \text{ind}(y)$, we have $\dim W^u(x) \cap W^s(y) = \text{ind}(x) - \text{ind}(y)$.

In particular if $\text{ind}(y) = \text{ind}(x) - 1$ then $\dim W^u(x) \cap W^s(y) = 1$. Consider a regular level set $f^{-1}(a)$, which is oriented. Then $W^u(x) \cap f^{-1}(a)$ and $W^s(x) \cap f^{-1}(a)$ intersect transversally and of complement dimensions. Thus in this case there are only a finite number of flow lines from x to y. We

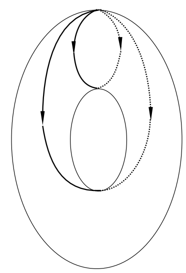

FIGURE 2. Flow lines on the torus.

assign a sign to each trajectory ϕ connecting x anf y as follows. Let $z \in \phi \cap f^{-1}(a)$. If the pair $(\mathrm{T}_z(W^u(x) \cap f^{-1}(a)), \mathrm{T}_z(W^s(y) \cap f^{-1}(a)))$ gives the same orientation as that of $f^{-1}(a)$ then assign the sign 1 to ϕ, and assign -1 otherwise. Then we denote by (x, y) the algebraic count of the trajectories from x to y.

Now we form the Morse complex (C, ∂) with chain groups C_i to be the free $\mathbb{Z}$-module generated by the critical points of index i. The boundary map ∂ is defined as:

$$\partial : C_i \longrightarrow C_{i-1}$$
$$\partial(x) = \sum_{\mathrm{ind} y = i-1} (x, y) y,$$

the sum is taken over all critical points y of index $i - 1$.

Theorem 2.1. *(C, ∂) is a complex and its homology is isomorphic to the singular homology of M*

$$H_*(C_*, \partial) \cong H_*(M, \mathbb{Z}).$$

2.2. **Circle-valued Morse theory.** In the early 1980's S.P. Novikov[8] has generalized the above construction of the Morse complex from real-valued functions to differential closed 1-forms. Here we only describe the construction of the Novikov complex in the simplest case when the closed form arised from a function with values on a circle. For details see Pazhitnov[9].

Let $s : M \longrightarrow S^1$ be a smooth function. Let $\omega = ds$, then ω is a smooth closed 1-form on M. The zeros of ω are the critical points of s. Since ω is locally exact, locally it is the derivative of a real-valued function. We say that ω is a Morse form if its zero points are non-degenerate critical points of the above locally-defined funtions, and define the indexes of zero points as indexes of the critical points.

We have a map $[\omega] : \pi_1(M) \longrightarrow \mathbb{R}$ defined as $[\omega](c) := \int_c \omega$. We can consider $[\omega]$ as an element of $H^1(M, \mathbb{R})$. In what follows we will only consider rank-1 forms, i.e. forms ω for which $\pi_1(M)/\mathrm{Ker}([\omega]) = \mathrm{Im}([\omega])$ has rank 1 in $\mathbb{R}$.

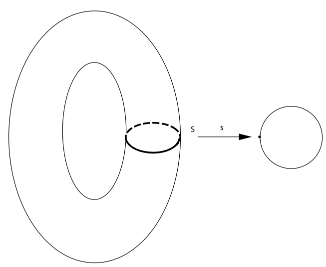

FIGURE 3. Circle-valued function and level set.

In that case there is a covering space $(\widetilde{M}, p)$ of M with infinite cyclic covering transformations group $\mathbb{Z} =< t >$ such that the pullback 1-form $p^*\omega$ is exact on $\widetilde{M}$, i.e. $p^*\omega = df$, where f is a smooth function from $\widetilde{M}$ to $\mathbb{R}$. (It can be checked that $p^*\omega$ is exact if and only if $p_*\pi_1(\widetilde{M})$ is a subset of $\mathrm{Ker}([\omega])$. Here $p_*\pi_1(\widetilde{M}) = \mathrm{Ker}([\omega])$.) The zeros of ω correspond to the critical points of f. Choose t so that $f(t \cdot x) < f(x)$.

As for the Morse complex, we consider the flow corresponds to the gradient vector field of f on $\widetilde{M}$. Orient the stable, unstable manifolds and trajectories as before. Consider two zeros x of index i and y of index $i - 1$ of ω. Choose arbitrary lifting points $\tilde{x}$ and $\tilde{y}$. There is only a finite number

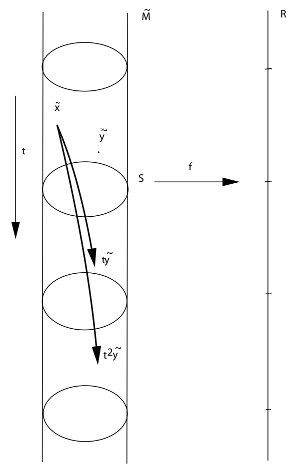

FIGURE 4. The cover $\widetilde{M}$ and flow lines.

of trajectories from $\tilde{x}$ to $t^k \cdot \tilde{y}$, for $k \in \mathbb{Z}$ (because these trajectories lie in a compact cobordism in $\widetilde{M}$). Denote by $(x, t^k \cdot y)$ the algebraic count of the oriented trajectories from $\tilde{x}$ to $t^k \cdot \tilde{y}$.

Introduce the *Novikov ring* $\mathbb{Z}((t)) = \mathbb{Z}[[t]][t^{-1}] = \{\sum_{n=-N}^{\infty} a_n t^n,\ a_n \in \mathbb{Z}\}$, the ring of Laurent power series with integer coefficients and finite negative degrees part. It is known that $\mathbb{Z}((t))$ is a principal ideal domain. We define the incidence coefficient to be $n(x, y) = \sum_k (x, t^k \cdot y) t^k$, we can see that $n(x, y) \in \mathbb{Z}((t))$.

Now we define the Novikov complex (CN_*, ∂). The chain group CN_i is the free $\mathbb{Z}((t))$-module generated by the zeros of index i of ω. The boundary map is defined as:

$$\partial : CN_i \longrightarrow CN_{i-1}$$
$$\partial(x) = \sum_{\mathrm{ind} y = i-1} n(x, y) y,$$

the sum is taken over all zero points y of index $i - 1$.

Theorem 2.2. *(CN_*, ∂) is a complex and*

$$H_*(CN_*) \cong H_*(C_*(\widetilde{M}) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}((t))),$$

here $C(\widetilde{M})$ denotes the cellular complex of $\widetilde{M}$.

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