

**REIDEMEISTER TORSION, TWISTED ALEXANDER
POLYNOMIAL, THE A -POLYNOMIAL, AND THE COLORED JONES
POLYNOMIAL OF SOME CLASSES OF KNOTS**

by

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Abstract

This dissertation studies invariants of knots and links.

In Chapter 1 we study a twisted Alexander polynomial of links in the projective space \mathbb{RP}^3 using its identification with Reidemeister torsion. We prove a skein relation for this polynomial.

Chapter 2 studies relationships between the A -polynomial of a 2-bridge knot and a twisted Alexander polynomial associated with the adjoint representation of the fundamental group of the knot complement. We show that for twist knots the A -polynomial is a factor of the twisted Alexander polynomial.

Chapter 3 studies the irreducibility of the A -polynomial of 2-bridge knots. We show that the A -polynomial $A(L, M)$ of a 2-bridge knot $\mathbf{b}(p, q)$ is irreducible if p is prime, and if $(p-1)/2$ is also prime and $q \neq 1$ then the L -degree of $A(L, M)$ is $(p-1)/2$. This shows that the AJ conjecture relating the A -polynomial and the colored Jones polynomial holds true for these knots, according to work of Le.

In Chapter 4 a determinant formula for the colored Jones polynomial is obtained. This determinant formula is similar to the known determinant formula for the volume of a hyperbolic knot obtained via L^2 -torsion. This study is in the context of the volume conjecture relating the colored Jones polynomial to the hyperbolic volume of a knot.

Major parts of this dissertation are joint works with Thang T. Q. Le.

CHAPTER 1

Twisted Alexander polynomial of links in \mathbb{RP}^3

1.1. Introduction

The study of polynomial invariants for links in the projective space \mathbb{RP}^3 was initiated in 1990 by Drobotukhina [Dro90]. She provided a set of Reidemeister moves for links in \mathbb{RP}^3 , and constructed an analogue of the Jones polynomial using Kauffman's approach involving state sum and the Kauffman bracket. Later she composed a table of links in \mathbb{RP}^3 up to six crossings, using the method of Conway's tangles [Dro94]. Recently Mroczkowski [Mro03b] defined the Homflypt and Kauffman polynomials using an inductive argument on descending diagrams similar to the argument for S^3 .

The twisted Alexander polynomial of a link associated to a representation of the fundamental group of the link's complement to $GL(n, \mathbb{C})$ is a generalization of the Alexander polynomial and has been studied since the early 1990s. It has been shown that in some circumstances the twisted polynomial is more powerful than the usual one: It could distinguish some pairs of knots which the usual polynomial could not, and it also provides more information on fiberedness and sliceness of knots.

For a link in \mathbb{RP}^3 , the first homology group of its complement has a torsion part. The Alexander polynomial will not detect information coming from this torsion part. In this chapter we will study a version of the twisted Alexander polynomial defined by Turaev, which takes the torsion part of the first homology group into account.

In his 1986 paper on applications of Reidemeister torsion in knot theory Turaev [Tur86] studied extensively the Alexander polynomial using the method of Reidemeister torsion. By introducing a refinement of Reidemeister torsion – the sign-refined

torsion – he was able to normalize the Alexander polynomial and derive a skein relation for it. Since then the sign-refined torsion has played important roles in such works as on the Casson invariant [Les96] and the Seiberg-Witten invariant [Tur01].

Here, following Turaev’s method, we identify our twisted Alexander polynomial with a corresponding Reidemeister torsion (Theorem 1.6.1). Using torsion we are able to derive a skein relation for the polynomial with a certain indeterminacy (Theorem 1.7.5). Then by introducing sign-refined torsion we normalized the twisted Alexander polynomial and provide a skein relation without indeterminacies (Theorem 1.7.8). Finally we study relationships between the twisted Alexander polynomial of a link and the Alexander polynomial of the link’s lift to S^3 (Theorem 1.8.3), also using Reidemeister torsion.

In our view the interest here lies primarily on the method. Skein relations for link polynomial invariants are usually studied diagrammatically on two-dimensional link projections. Here we study skein relations through three-dimensional topology, using a classical yet contemporary topological invariant – the Reidemeister torsion. As both the twisted Alexander polynomial and the Reidemeister torsion continue to be interesting subjects of studies (see e.g. Chapters 2 and 4) this point of view may have promises for future studies.

1.2. Background on Reidemeister torsion

The general references for this section are [Mil66] and [Tur01].

1.2.1. Torsion of a chain complex. Let \mathbb{F} be a field, V be a k -dimensional vector space over \mathbb{F} . Suppose that $b = (b_1, b_2, \dots, b_k)$ and $c = (c_1, c_2, \dots, c_k)$ are two ordered bases of V . Then there is a non-singular $k \times k$ matrix (a_{ij}) such that $c_j = \sum_{i=1}^k a_{ij} b_i$. We write $[c/b] = \det(a_{ij}) \in \mathbb{F}^* = \mathbb{F} \setminus \{0\}$. By linear algebra we have $[b/b] = 1$, and if d is another basis then $[d/b] = [d/c][c/b]$.

We call two bases b and c *equivalent* if $[b/c] = 1$. The above properties show that this is indeed an equivalence relation. We will identify a basis with its equivalence class. Also we say b and c have the same orientation if $[b/c] > 0$.

Let $0 \rightarrow C \xrightarrow{\alpha} D \xrightarrow{\beta} E \rightarrow 0$ be a short exact sequence of vector spaces. Then $\dim D = \dim C + \dim E$. Let $c = (c_1, c_2, \dots, c_k)$ be a basis for C and $e = (e_1, e_2, \dots, e_l)$ be a basis for E . Since β is surjective we can lift e_i to a vector \tilde{e}_i in D . Then using linear algebra it can be proved that $ce = (c_1, \dots, c_k, \tilde{e}_1, \dots, \tilde{e}_l)$ is a basis for D and its equivalence class depend not on the choice of \tilde{e}_i but only on the equivalence classes of c and e .

The finite chain complex $(C, \partial) = (0 \rightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0)$ of finite-dimensional vector spaces over \mathbb{F} is called *acyclic* if it is exact. In that case $H_*(C) = 0$. The chain is called *based* if for each C_i a basis is chosen.

Assume that (C, ∂) is acyclic and based with basis c . Let $B_i = \text{Im}(\partial_{i+1} : C_{i+1} \rightarrow C_i) \subset C_i$. Choose a basis b_i for B_i . We have $C_i / \ker \partial_i = \text{Im} \partial_i = B_{i-1}$. Since $\ker \partial_i = \text{Im} \partial_{i+1} = B_i$, we get $C_i / B_i = B_{i-1}$. In other words we have the short exact sequence $0 \rightarrow B_i \hookrightarrow C_i \twoheadrightarrow B_{i-1} \rightarrow 0$. By the above argument $b_i b_{i-1}$ is a basis for C_i . But C_i already has a basis c_i .

DEFINITION 1.2.1. The torsion of the acyclic and based chain complex C is defined to be

$$\tau(C) = \prod_{i=0}^m [b_i b_{i-1} / c_i]^{(-1)^{i+1}} \in \mathbb{F}^*.$$

If C is not acyclic then $\tau(C)$ is defined to be 0.

Note that this torsion (Turaev's version) is the inverse of Milnor's version.

The torsion $\tau(C)$ depends on the basis c but does not depend on the choice of the bases b_i 's. If we use a different basis c'_i instead of c_i for C_i then the torsion is

multiplied with

$$(1.2.1) \quad [c_i/c'_i]^{(-1)^{i+1}}.$$

REMARK 1.2.2. In computations it is convenient to write $\tau(C)$ as a product of terms of the form $[(\partial_{i+1}(c_{i+1}), \widetilde{\partial_i(c_i)})/c_i]^{(-1)^{i+1}}$.

EXAMPLE 1.2.3. a). If $C = (0 \rightarrow C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0)$ is acyclic then $\tau(C) = \det^{-1}(\partial_1)$.

b). If $C = (0 \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0)$ is acyclic and \tilde{c}_0 is a lift of c_0 to C_1 then $\tau(C) = [(\partial_2 c_2)\tilde{c}_0/c_1]$.

1.2.1.1. *Change of rings.* Let R be a ring and (C, ∂) be a based chain complex of free finitely generated left modules over R . Suppose that $\varphi : R \rightarrow \mathbb{F}$ is a ring homomorphism. Then by using φ we can consider \mathbb{F} as a right R -module, namely for $r \in R$ and $a \in \mathbb{F}$ we define $a \cdot r = a\varphi(r)$. Thus we can form the tensor $\mathbb{F} \otimes_{\varphi} C_i$ of modules over \mathbb{F} . Moreover $\mathbb{F} \otimes_{\varphi} C_i$ is a vector space over \mathbb{F} , and if $\{e_1, e_2, \dots, e_n\}$ is a basis for C_i then $\{1 \otimes e_1, 1 \otimes e_2, \dots, 1 \otimes e_n\}$ is a basis for $\mathbb{F} \otimes_{\varphi} C_i$, where 1 is the unit of \mathbb{F} . The boundary map of the chain complex C induces the boundary map for the chain $(\mathbb{F} \otimes_{\varphi} C_*)$. Thus $(\mathbb{F} \otimes_{\varphi} C_*)$ becomes a based chain complex of finite-dimensional vector spaces over \mathbb{F} . If it is acyclic then we can define its torsion $\tau(\mathbb{F} \otimes_{\varphi} C_*) \in \mathbb{F}^*$.

1.2.2. Torsion of a CW-complex. Let X be a finite connected CW-complex. The universal cover \tilde{X} of X has a canonical CW-complex structure obtained by lifting the cells of X . The group $\pi = \pi_1(X)$ of covering transformations of \tilde{X} acts freely and transitively on \tilde{X} . Moreover, if \tilde{e} and \tilde{e}' are two liftings of the same cell e of X then there is a unique element of π that maps \tilde{e} to \tilde{e}' . Thus this action of π on \tilde{X} can be extended canonically to an action of π on the cellular chain groups $C_i(\tilde{X})$. Then we extend this action linearly to a $\mathbb{Z}[\pi]$ -action on $C_i(\tilde{X})$, and so $C_i(\tilde{X})$ becomes a (left)

$\mathbb{Z}[\pi]$ -module. If $\{e_i^k, 1 \leq i \leq n_k\}$ is an ordered set of oriented k -cells of X and \tilde{e}_i^k is any lift of e_i^k then the ordered set $\{\tilde{e}_i^k, 1 \leq i \leq n_k\}$ is a basis of the $\mathbb{Z}[\pi]$ -module $C_i(\tilde{X})$.

Suppose that \mathbb{F} is a field and $\mathbb{Z}[\pi] \xrightarrow{\varphi} \mathbb{F}$ is a ring homomorphism. Then by the change of rings construction in the previous section, $\mathbb{F} \otimes_{\varphi} C_*(\tilde{X})$ becomes a chain complex of finite dimensional vector spaces over \mathbb{F} . If this chain complex is acyclic then we can define its torsion $\tau(\mathbb{F} \otimes_{\varphi} C_*(\tilde{X})) \in \mathbb{F}^*$. However $\tau(\mathbb{F} \otimes_{\varphi} C_*(\tilde{X}))$ depends on the chosen basis for $C_*(\tilde{X})$, that is on the choices of lifting cells $\{\tilde{e}_i^k, 1 \leq i \leq n_k\}$ made above. If we fix a choice of a set of lifting cells as a basis for the $\mathbb{Z}[\pi]$ -module $C_i(\tilde{X})$ but change the order of the cells in the basis then by Formula (1.2.1) $\tau(\mathbb{F} \otimes_{\varphi} C_*(\tilde{X}))$ is multiplied with ± 1 . If we change the orientations of the cells then torsion is also multiplied with ± 1 . If we choose a different lifting cell for e_i^k – by an action $h \cdot \tilde{e}_i^k$ of a covering transformation $h \in \pi$ – then the torsion is multiplied with $\varphi(h)^{\pm 1}$.

DEFINITION 1.2.4. The Reidemeister torsion $\tau^{\varphi}(X)$ of the CW-complex X is defined to be the image of $\tau(\mathbb{F} \otimes_{\varphi} C_*(\tilde{X}))$ under the quotient map $\mathbb{F} \rightarrow \mathbb{F}/\pm\varphi(\pi)$.

1.2.3. Topological invariance of torsion, and examples. At the time of [Mil66] it was already known that the torsion of a CW-complex is invariant under cellular subdivisions. Torsion is proved to be a simple homotopy invariant, by showing that it does not change under elementary collapsings or expansions [Tur01, p. 43]. However in general torsion is not a homotopy invariant. Chapman (1974) proved that torsion is a topological invariant (i.e. a homeomorphism invariant) of compact connected CW-complex.

In dimensions three or less, which is where our main interests are, each topological manifold has a unique piecewise-linear structure, so the torsion of the manifold can be defined.

Torsion played important roles in some of the major results of classical topology in the 1960's, for example in the h-cobordism theorem ([**RS72**, p. 88]), and in works related to the Hauptvermutung (e.g. [**Mil61**]).

EXAMPLE 1.2.5 (Torsion of the circle). The circle has a cell decomposition consists of one 0-cell e^0 and one 1-cell e^1 , and $\pi_1(S^1) = \langle t \rangle$. Its universal cover is \mathbb{R} , with the induced cell structure. Consider the boundary map $\partial_1 : C_1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$. We can choose the lifts of e^1 and e^0 so that $\partial_1(\tilde{e}^1) = (t - 1)\tilde{e}^0$. As a homomorphism between one-dimensional $\mathbb{Z}[t^{\pm 1}]$ -modules, ∂_1 is represented by the matrix $\partial_1 = [(t - 1)]$. The chain complex $0 \rightarrow \mathbb{F} \otimes_{\varphi} C_1(\mathbb{R}) \xrightarrow{\partial_1} \mathbb{F} \otimes_{\varphi} C_0(\mathbb{R}) \rightarrow 0$ is acyclic if and only if $\partial_1 = [\varphi(t - 1)]$ is bijective, that is if and only if $\varphi(t - 1)$ is invertible. We use a ring homomorphism φ to map $\mathbb{Z}[t^{\pm 1}]$ to a field \mathbb{F} so that the image of $t - 1$ is invertible in \mathbb{F} . For example, \mathbb{F} can be $\mathbb{Z}(t) = Q(\mathbb{Z}[t^{\pm 1}])$ with φ being the canonical imbedding, or $\mathbb{F} = \mathbb{C}$ with φ sending t to $\xi \neq 1$. Then $\tau^{\varphi}(S^1) = \det^{-1}(\partial_1) = (\varphi(t) - 1)^{-1} \in \mathbb{F}/\pm\varphi(t)^n$, $n \in \mathbb{Z}$.

EXAMPLE 1.2.6 (Torsion of the torus). The torus T^2 has a cell decomposition consists of one 0-cell e^0 , two 1-cells e_1^1 and e_2^1 , and a 2-cell e^2 . Its universal cover, which is also its maximal abelian cover, is \mathbb{R}^2 with the induced cell decomposition, and the covering transformation group is $\pi_1(T^2) = \langle t_1 \rangle \oplus \langle t_2 \rangle$. The cellular chain complex of \mathbb{R}^2 is a chain complex of $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$ -modules: $0 \rightarrow C_2(\mathbb{R}^2) \xrightarrow{\partial_2} C_1(\mathbb{R}^2) \xrightarrow{\partial_1} C_0(\mathbb{R}^2) \rightarrow 0$. The boundary maps are given by $\partial_2 = (1 - t_2, t_1 - 1)^t$ and $\partial_1 = (t_1 - 1, t_2 - 1)$.

Apply the change of ring construction associated with a ring homomorphism $\varphi : \mathbb{Z}[\pi_1(T^2)] \rightarrow \mathbb{F}$ we obtain the chain complex of \mathbb{F} -vector spaces: $0 \rightarrow \mathbb{F} \otimes_{\varphi} C_2(\mathbb{R}^2) \xrightarrow{\partial_2} \mathbb{F} \otimes_{\varphi} C_1(\mathbb{R}^2) \xrightarrow{\partial_1} \mathbb{F} \otimes_{\varphi} C_0(\mathbb{R}^2) \rightarrow 0$, where the boundary maps are $\partial_2 = \begin{pmatrix} 1 - \varphi(t_2) \\ \varphi(t_1) - 1 \end{pmatrix}$ and $\partial_1 = \begin{pmatrix} \varphi(t_1) - 1 & \varphi(t_2) - 1 \end{pmatrix}$. It is easy to check directly that the chain is exact if and only if either $(\varphi(t_1) - 1) \neq 0$ or $(\varphi(t_2) - 1) \neq 0$. Choose the lifting cells so that they have a

common intersection at the lift of the zero cell. Then the torsion $\tau^\varphi(T^2)$ is

$$\begin{aligned} [(\partial_2 \tilde{e}^2, \tilde{e}^0)/(\tilde{e}_1^1, \tilde{e}_2^1)] &= [((1 - \varphi'(t_2))\tilde{e}_1^1 + (\varphi(t_1) - 1)\tilde{e}_2^1, \frac{1}{\varphi(t_1) - 1}\tilde{e}_1^1)/(\tilde{e}_1^1, \tilde{e}_2^1)] \\ &= \det \begin{pmatrix} 1 - \varphi(t_2) & \frac{1}{\varphi(t_1) - 1} \\ \varphi(t_1) - 1 & 0 \end{pmatrix} = -1, \end{aligned}$$

up to $\pm\varphi(t_1)^m\varphi(t_2)^n$; $m, n \in \mathbb{Z}$, depending on how the lifting cells are chosen.

1.2.4. Torsion with homological bases. Here we consider the case when the chain complex is not acyclic, following [Mil66, p. 158]. Suppose that $(C, \partial) = (0 \rightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0)$ is a chain complex of based finite-dimensional vector spaces, not necessarily acyclic. Let $Z_i = \ker \partial_i$ and $B_i = \text{Im } \partial_{i+1}$. Let $H_i(C) = Z_i/B_i$ be the i th homology group and h_i be its chosen basis. There is a short exact sequence $0 \rightarrow B_i \hookrightarrow Z_i \twoheadrightarrow H_i \rightarrow 0$. This combined with the short exact sequence $0 \rightarrow Z_i \hookrightarrow C_i \twoheadrightarrow B_{i-1} \rightarrow 0$ show that $(b_i h_i) b_{i-1}$ is a basis for C_i (and is defined up to equivalence bases). We can define torsion in a similar manner:

$$(1.2.2) \quad \tau(C, h) = \prod_{i=0}^m [b_i h_i b_{i-1} / c_i]^{(-1)^{i+1}} \in \mathbb{F}^*.$$

The torsion $\tau(C, h)$ does not depend on the choice of the bases b_i 's, however it depends on c and h .

1.2.5. Symmetry of torsion. Here we follow [Tur01, p. 70]. Let M be a compact connected three-manifold with or without boundary. Let $\sigma : \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[\pi]$ be the involution defined by $\sigma(\alpha) = (-1)^{\omega_1(\alpha)} \alpha^{-1}$ for $\alpha \in \pi$, where $\omega_1 : \pi \rightarrow \mathbb{Z}_2$ is the first Stiefel–Whitney class of M . Recall that $\omega_1(\alpha)$ is 0 if α is orientation preserving and 1 otherwise, so if M is orientable then ω_1 is identical to zero and $\sigma(\alpha) = \alpha^{-1}$. Let $\varphi : \mathbb{Z}[\pi] \rightarrow \mathbb{F}$ be a ring homomorphism.

THEOREM 1.2.7. *If $H_*^\varphi(M) = 0$ then $H_*^{\varphi\circ\sigma}(M, \partial M) = 0$ and $\tau^{\varphi\circ\sigma}(M, \partial M) = \tau^\varphi(M)$.*

Suppose that in the field \mathbb{F} there is a certain “bar” operation so that for all $\alpha \in \pi$, $\overline{\varphi(\alpha)} = (-1)^{\omega_1(\alpha)}\varphi(\alpha^{-1})$. For example in the case of orientable manifolds we can take $\bar{a} = a^{-1}$ for $a \in \mathbb{F}$. Then $\bar{\varphi} = \varphi \circ \sigma$. The theorem above now gives (using functority of torsion) $\overline{\tau^\varphi(M, \partial M)} = \tau^\varphi(M)$.

The following result essentially comes from the fact that torsion of a torus is 1:

PROPOSITION 1.2.8. *If ∂M consists of tori then $\tau^\varphi(M, \partial M) = \tau^\varphi(M)$.*

The following theorem is now immediate:

THEOREM 1.2.9 (Symmetry of torsion). *If ∂M consists of tori then $\tau^\varphi(M) = \overline{\tau^\varphi(M)}$.*

1.2.6. Sign-refined torsion. This was introduced by Turaev [Tur86] to remove the sign ambiguity of torsion. Let C be a finite based chain complex of vector spaces over a field \mathbb{F} . Let $\beta_i(C) = \sum_{j=0}^i \dim(H_j(C)) \pmod{2}$, $\gamma_i(C) = \sum_{j=0}^i \dim(C_j) \pmod{2}$, and $N(C) = \sum \beta_i(C)\gamma_i(C) \pmod{2}$. Let c be a basis for C_* and h be a basis for $H_*(C)$. Define

$$(1.2.3) \quad \check{\tau}(C, c, h) = (-1)^{N(C)}\tau(C, c, h) \in \mathbb{F}.$$

Thus $\check{\tau}(C, c, h)$ is $\tau(C, c, h)$ up to a sign, and they are the same when C is acyclic.

A *homological orientation* for a finite CW-complex X is an orientation of the finite dimensional vector space $\oplus_i H_i(X, \mathbb{R})$. Let h be a basis for $H_*(X, \mathbb{R})$ representing a homological orientation, i.e. h is a positive basis, and let c be a basis for $C_*(X, \mathbb{Z})$ arising from an ordered set of oriented cells of X , which gives rise to a basis for $C_*(X, \mathbb{R})$. We call a lift \tilde{c} of c to the universal cover \tilde{X} a *fundamental family of cells*.

Let

$$(1.2.4) \quad \tau_0^\varphi(X, \tilde{c}, h) = \text{sign}\left(\check{\tau}(C_*(X; \mathbb{R}), c, h)\right) \tau^\varphi(X, \tilde{c}).$$

DEFINITION 1.2.10. The sign-defined torsion of the finite connected CW-complex X with a homological orientation represented by h is the image of $\tau_0^\varphi(X, \tilde{c}, h)$ under the projection $\mathbb{F} \rightarrow \mathbb{F}/\varphi(\pi_1(X))$.

This torsion has no sign ambiguity. It depends on the homological orientation class of represented by h but not on the order or the orientations of the cells of X . The choice of the number $N(C)$ results from a change of base formula. With this number the sign-refined torsion is invariant under simple homotopy equivalences preserving homology orientations. Because homeomorphisms of finite connected CW-complex are simple homotopy equivalences, the torsion is a homeomorphism invariant, in particular it is invariant under cellular subdivisions [Tur01, p. 98].

1.2.7. Product formulas.

1.2.7.1. *Product formulas for unrefined torsion.* Suppose that $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ is a short exact sequence of finite acyclic chain complex of vector spaces. Suppose that the bases of C , C' and C'' are *compatible*, in the sense that c_i is equivalent to $c'_i c''_i$, then

$$(1.2.5) \quad \tau(C) = \pm \tau(C') \tau(C'').$$

When the chains are not acyclic there is also a product formula for torsion with homological bases. Let h , h' and h'' be the bases for $H_*(C)$, $H_*(C')$ and $H_*(C'')$ respectively. The short exact sequence involving C , C' , C'' above gives rise to a finite long exact sequence of homology groups $\mathcal{H} = (\cdots \rightarrow H_i(C') \rightarrow H_i(C) \rightarrow H_i(C'') \rightarrow H_{i-1}(C') \rightarrow \cdots)$. Since these vector spaces are based the chain \mathcal{H} has a well-defined torsion $\tau(\mathcal{H})$, which depends on h , h' and h'' . Suppose that the bases of C , C' and

C'' are compatible, then

$$(1.2.6) \quad \tau(C, h) = \pm \tau(C', h') \tau(C'', h'') \tau(\mathcal{H}).$$

1.2.7.2. *Product formulas for refined torsion.* First observe this product formula for chain complexes of vector spaces, in which the work of keeping track of the shuffling of the bases is done (cf. [Tur86, Lemma 3.4.2]).

$$(1.2.7) \quad \check{\tau}(C, c', h) = (-1)^{\mu+\nu} \check{\tau}(C', c', h') \check{\tau}(C'', c'', h'') \tau(\mathcal{H}),$$

in other words

$$(1.2.8) \quad \tau(C, c'c'', h) = (-1)^{\mu+\nu+N(C)+N(C')+N(C'')} \tau(C', c', h') \tau(C'', c'', h'') \tau(\mathcal{H}),$$

where $\mu = \sum[(\beta_i(C) + 1)(\beta_i(C') + \beta_i(C'')) + \beta_{i-1}(C')\beta_i(C'')] \pmod 2$; and $\nu = \sum_{i=0}^m \gamma_i(C'')\gamma_{i-1}(C')$.

Let (X, Y) be a CW-complex pair. Suppose that $C_*(Y; \mathbb{R})$ and $C_*(X, Y; \mathbb{R})$ have homological orientations with bases h and h' so that the torsion of the long exact induced homological sequence of the pair is positive. This condition determine a homological orientation for $C_*(X; \mathbb{R})$, denoted by hh' . Let c and c' be fundamental family of cells for Y and (X, Y) , there is a canonical fundamental family of cells for X , denoted by cc' . If either $\tau^\varphi(X, Y) \neq 0$ or $\tau^\varphi(Y) \neq 0$ then

$$(1.2.9) \quad \tau^\varphi(X, cc', hh') = (-1)^\mu \tau^\varphi(Y, c, h) \tau^\varphi(X, Y, c', h'),$$

where $\mu = \sum[(\beta_i(X) + 1)(\beta_i(Y) + \beta_i(X, Y)) + \beta_{i-1}(Y)\beta_i(X, Y)] \pmod 2$.

1.2.8. Homological orientations of oriented link complements. Suppose that L is an oriented link in an oriented rational homology three-sphere M , and let X be the link complement. We want to determine the homology groups with real coefficients of X . Let $U = N(L)$. The Mayer-Vietoris for the pair (X, U) with real

coefficients gives:

$$\begin{aligned} 0 \rightarrow H_3(M) \xrightarrow{\Delta} H_2(U \cap X) \xrightarrow{(i_*, j_*)} H_2(U) \oplus H_2(X) \rightarrow 0 \rightarrow H_1(U \cap X) \xrightarrow{(i_*, j_*)} \\ \xrightarrow{(i_*, j_*)} H_1(U) \oplus H_1(X) \rightarrow 0 \rightarrow H_0(U \cap X) \rightarrow H_0(U) \oplus H_0(X) \rightarrow H_0(M) \rightarrow 0, \end{aligned}$$

where i_* and j_* are induced from the inclusions of $U \cap X$ to U and X respectively.

In dimension zero $H_0(X, \mathbb{R})$ is of course \mathbb{R} , generated by a point. The first dimension is also simple, we have $0 \rightarrow \sum_{1 \leq i \leq v} \mathbb{R}[m_i] \oplus \mathbb{R}[l_i] \xrightarrow{(i_*, j_*)} H_1(U) \oplus H_1(X) \rightarrow 0$, where m_i and l_i are the meridian and the longitude of the component ∂U_i , $1 \leq i \leq v$. Thus $H_1(X; \mathbb{R}) = \sum_{1 \leq i \leq v} \mathbb{R}[m_i]$.

Consider the second dimension. We have $0 \rightarrow \mathbb{R}[M] \xrightarrow{\Delta} \sum_{1 \leq i \leq v} \mathbb{R}[\partial U_i] \xrightarrow{(i_*, j_*)} H_2(X) \rightarrow 0$. Since $M = X \cup_i U_i$ we have $[M] = [X] + \sum_i [U_i]$ and $[\partial X] + \sum_i [\partial U_i] = 0$. According to Mayer-Vietoris, $\Delta([M])$ is just $[\partial X] = -\sum_i [\partial U_i]$. Thus $H_2(X; \mathbb{R}) \cong (\sum_{1 \leq i \leq v} \mathbb{R}[\partial U_i]) / (\mathbb{R} \sum_i [\partial U_i]) \cong \sum_{1 \leq i \leq v-1} \mathbb{R}[\partial U_i]$.

The canonical homological orientation of the oriented link L is the orientation of the vector space $H_*(X; \mathbb{R})$ represented by the basis $([pt], [m_1], \dots, [m_v], [\partial U_1], \dots, [\partial U_{v-1}])$, i.e. this basis is declared to be positive. The classes $[m_i]$ depends on the orientation of L and so does the homological orientation.

1.2.9. Reidemeister torsion associated with representations to $\mathrm{SL}(n, \mathbb{C})$.

The Reidemeister torsion associated with a representation to $\mathrm{O}(n)$ was considered by Milnor [Mil66, p. 180]. Here we follow Kitano's treatment in [Kit96]. Let X be a finite connected CW-complex and $\pi = \pi_1(X)$. With respect to the universal covering \tilde{X} we construct a left $\mathbb{Z}[\pi]$ -module $C_*(\tilde{X})$. Let $\rho : \pi \rightarrow \mathrm{SL}(n, \mathbb{C})$ be a representation. There is a natural action of $\mathrm{SL}(n, \mathbb{C})$ on \mathbb{C}^n , which is the right multiplication of a matrix with a vector. Similarly to the change of rings process described in Section 1.2.1.1, using ρ we can view \mathbb{C}^n as a right $\mathbb{Z}[\pi]$ -module. Thus we can form the tensor product $C_i^\rho(X) = \mathbb{C}^n \otimes_{\mathbb{Z}[\pi], \rho} C_i(\tilde{X})$, which is a vector space over \mathbb{C} . Let $\{e_1, e_2, \dots, e_n\}$

be the standard basis of \mathbb{C}^n and $\{\sigma_1^i, \sigma_2^i, \dots, \sigma_{k_i}^i\}$ be the ordered set of i -cells of X then we have a basis for the vector space $C_i^\rho(X)$ as $\{e_1 \otimes \sigma_1^i, e_2 \otimes \sigma_1^i, \dots, e_n \otimes \sigma_1^i, e_1 \otimes \sigma_2^i, e_2 \otimes \sigma_2^i, \dots, e_n \otimes \sigma_2^i, e_1 \otimes \sigma_{k_i}^i, e_2 \otimes \sigma_{k_i}^i, \dots, e_n \otimes \sigma_{k_i}^i\}$. If the complex $C_*^\rho(X)$ is acyclic we can define the torsion $\tau^\rho(X) = \tau(C_*^\rho(X)) \in \mathbb{C}$. Because $\rho(\pi_1) \subset \mathrm{SL}(n, \mathbb{C})$ the determinant computations will destroy any ambiguity about the choice of representing cells, so $\tau^\rho(X)$ is defined up to ± 1 .

1.3. Diagrams for links in \mathbb{RP}^3

We follow the terminology of Drobotukhina in [Dro90]. Consider the standard model of \mathbb{RP}^3 as a ball B^3 with antipodal points on the boundary sphere ∂B^3 identified. In this way $\mathbb{RP}^3 = \mathbb{RP}^2 \cup B^3$. Let N and S be respectively the North Pole and the South Pole of ∂B^3 . Given a link L in \mathbb{RP}^3 , let \tilde{L} be its inverse image in B^3 under the quotient map. Isotope L a bit so that \tilde{L} does not pass through N or S . Define a projection map p from \tilde{L} to the equator disk D^2 , where a point x is mapped to the point $p(x)$ which is the intersection between the disk D^2 and the semicircle passing through three points N , S , and x . We also orient each of such semicircles in the direction from N to S .

We can always isotope L so that \tilde{L} satisfies the following conditions of general position:

- (1) \tilde{L} intersects the boundary sphere ∂B^3 transversally, no two points of \tilde{L} lies on the same half of a great circle joining N and S (i.e. $p(\tilde{L})$ has no double point on the boundary circle ∂D^2).
- (2) The projection $p(\tilde{L})$ contains no cusps, no points of tangency, and no triple points.

At each double point P of $p(\tilde{L})$, the inverse image $p^{-1}(P)$ consists of two points in \tilde{L} . The two points are on the same semicircle joining N and S . Since we have oriented this semicircle from N to S , the point nearer to N is called the upper point,

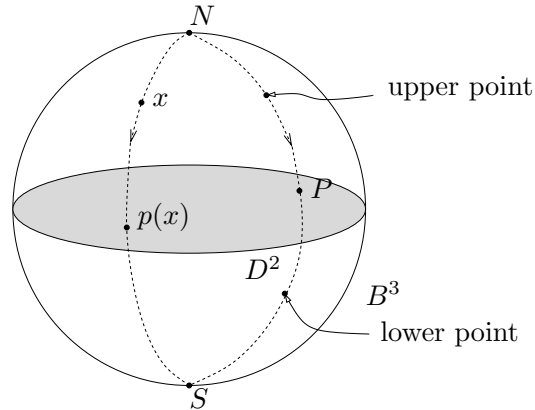


FIGURE 1.1. Standard model.

the other point is called the lower point. The projection of a small arc of \tilde{L} around an upper point is called an overpass, similarly, the projection of a small arc of \tilde{L} around a lower point is called an underpass. The projection $p(\tilde{L})$ together with information about overpasses and underpasses is called the diagram of the link L .

1.4. The fundamental group

1.4.1. A Wirtinger-type presentation for the fundamental group. Let D be a diagram of a link L (a knot is a link having one component). Choose an orientation for D . Label the upper arcs of D , each of which connecting two crossings of D as underpasses, as $a_1, a_2, \dots, a_q, 0 \leq q$, in arbitrary order (in case of an unknotting component which does not cross under, consider the whole component as an upper arc). Associate with each upper arc a_i a simple closed curve, also called a_i (the intended object is always clear from the context), starting at N , winding once around the arc, oriented following the right-hand screw rule. Let $2p, p \geq 0$ be the number of intersections between D and the boundary circle of the projection disk. Label the intersection point counterclockwise as b_1, b_2, \dots, b_{2p} , starting from any point. Associate with each point b_i a simple closed curve starting at N , winding once around a short arc of D connected to the point b_i , oriented following the right-hand screw rule. We

will also use the notation b_i to denote this simple closed curve. For each b_i , associate a number ϵ_i as follows. At the point b_i , if D is entering the boundary then let $\epsilon_i = 1$, and let $\epsilon_i = -1$ in the other case. Finally, let c be a simple arc running from N to S , intersecting the equator at a point between b_{2p} and b_1 . Compare with Figure 1.2.

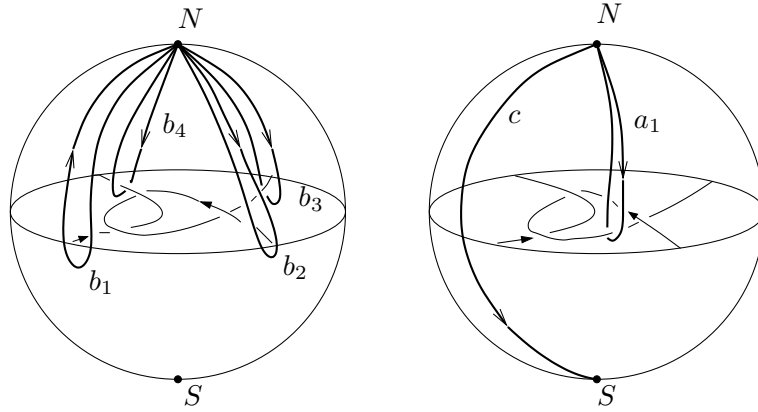


FIGURE 1.2. Generators.

THEOREM 1.4.1. *With the above notations $\pi_1(\mathbb{RP}^3 \setminus L)$ has a presentation with generators $a_1, a_2, \dots, a_q, b_1, b_2, \dots, b_{2p}, c$; and relations:*

$$b_{p+i} = c^{-1} b_1^{\epsilon_1} b_2^{\epsilon_2} \cdots b_{i-1}^{\epsilon_{i-1}} b_i b_{i-1}^{-\epsilon_{i-1}} b_{i-2}^{-\epsilon_{i-2}} \cdots b_1^{-\epsilon_1} c, 1 \leq i \leq p;$$

$$b_1^{\epsilon_1} b_2^{\epsilon_2} \cdots b_p^{\epsilon_p} = c^2$$

together with Wirtinger-type relations involving a_i and b_j at each crossing; and if there is an upper arc connecting b_i and b_j then there is a relation $b_i = b_j$.

PROOF. The proof is an application of the van Kampen theorem. Again, think of \mathbb{RP}^3 as a union of \mathbb{RP}^2 and a ball B^3 . Inside this B^3 , take a smaller open 3-ball B containing almost the entire link, including all the crossings, excepts the $2p$ small arcs connected to \mathbb{RP}^2 . To use N as the base point for fundamental groups, let T be a small open tubular neighborhood of an arc connecting N and B . Let U be $(B \setminus \overline{N(L)}) \cup T$, where $N(L)$ is an open neighborhood of L . Let $V = (\mathbb{RP}^3 \setminus \overline{N(L)}) \setminus \bar{B}$, thickened a bit

into U . Then both U and V are open and path-connected. The intersection $U \cap V$ is a thickened 2-sphere with punctures, namely it is $((S^2 \setminus \{2p \text{ points}\}) \times (-1, 1)) \cup T$, which is path-connected.

Computing $\pi_1(U)$. Following a proof of the Wirtinger presentation for links in S^3 , we would get that $\pi_1(U)$ is generated by $b_i, 1 \leq i \leq 2p$ and $a_j, 1 \leq j \leq q$ with Wirtinger-type relations at each crossing, and if there is an upper arc connecting b_i and b_j then we would have $b_i = b_j$.

Computing $\pi_1(U \cap V)$. Since homotopically $U \cap V$ is a $2p$ -punctured 2-sphere, its fundamental group is free with $2p - 1$ generators among b_1, b_2, \dots, b_{2p} .

Computing $\pi_1(V)$. Homotopically V is a p -punctured $\mathbb{R}P^2$. Its double cover \tilde{V} is a $2p$ -punctured S^2 , with the projection map being the map identifying antipodal points. Any loop in V which is based at N is lifted to \tilde{V} as either a loop based at N or an arc connecting N and S . If two loops in V are homotopic then their lifts are homotopic. Thus $\pi_1(V)$ is isomorphic to the group of homotopy classes of liftings in \tilde{V} of loops in V . Then $\pi_1(V) = \langle b_1, b_2, \dots, b_p, c / b_1^{\epsilon_1} b_2^{\epsilon_2} \cdots b_p^{\epsilon_p} = c^2 \rangle$.

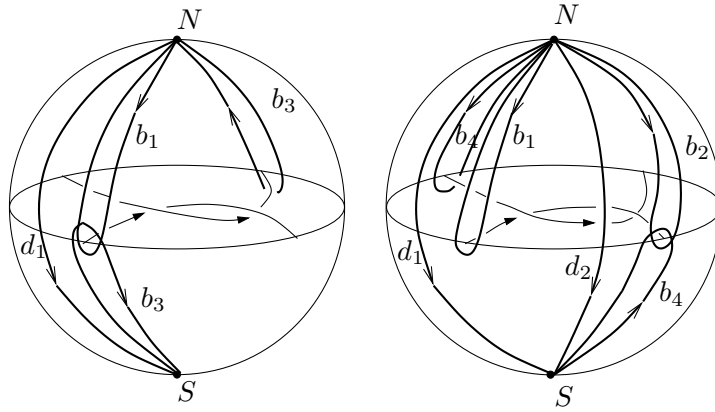


FIGURE 1.3. Relations.

Let $d_1 = c$ and $d_i, 2 \leq i \leq p$ be a simple arc running from N to S , passing through the equator at a point between b_i and b_{i+1} . The inclusions of $\pi_1(U \cap V)$ to $\pi_1(U)$ and

$\pi_1(V)$ give rise to the relations (compositions of paths are read from left to right)

$$b_{p+i} = d_i^{-1} b_i d_i, 1 \leq i \leq p,$$

where $d_{i-1} d_i^{-1} = b_{i-1}^{\epsilon_{i-1}}, 2 \leq i \leq p$, thus

$$d_i = b_{i-1}^{-\epsilon_{i-1}} d_{i-1} = b_{i-1}^{-\epsilon_{i-1}} b_{i-2}^{-\epsilon_{i-2}} \cdots b_1^{-\epsilon_1} c, 2 \leq i \leq p.$$

□

1.4.2. Presentations of deficiency one.

ASSERTION 1.4.2. *Assuming that the diagram D does not contain any unknotted (connected) component. Then its number of crossings is exactly $p + q$.*

PROOF. For the purpose of this proof imagine for a moment that each intersection point between D and the boundary is a fake crossing where the diagram D goes under. That means each arc a_i and b_j is an upper arc connecting two different (real or fake) crossings (this is not the case if there is an unknot component—hence the assumption). As we travel the diagram D starting from a base-point, following a given direction, it's clear that the total number of (real or fake) crossings is equal to the total number of arcs a_i and b_j , which is $2p + q$. There are p fake crossings, so there are exactly $p + q$ real crossings. □

ASSERTION 1.4.3. *Suppose that D contains no upper arc connecting two non-antipodal points and no affine unknot component. Then in the standard presentation of Theorem 1.4.1, the number of generators and the number of relations are the same.*

PROOF. When D does not have any upper arc connecting two antipodal points, i.e. D has no projective unknot component then using Assertion 1.4.2, the number of relations is $p + 1 + (p + q) = 2p + q + 1$, which is exactly the number of generators.

Suppose that we add to a given diagram an upper arc connecting two antipodal points b_i and b_j . If this new upper arc intersects the former diagram at k points then in the presentation associated with the new diagram we have $k + 2$ more generators. There are k new relations at each new crossing, plus a relation expressing b_j in terms of other generators, plus the relation $b_i = b_j$. Thus we have also added $k + 2$ more relations. □

ASSERTION 1.4.4. *A diagram can always be isotoped so that there is no upper arc connecting two non-antipodal points.*

PROOF. Suppose that there is such an upper arc with two end points A_1 and A_2 on the boundary circle. There are two corresponding arcs B_1B_2 and C_1C_2 . If the points B_1 and C_1 are close enough to the boundary then B_1B_2 and C_1C_2 are under arcs.

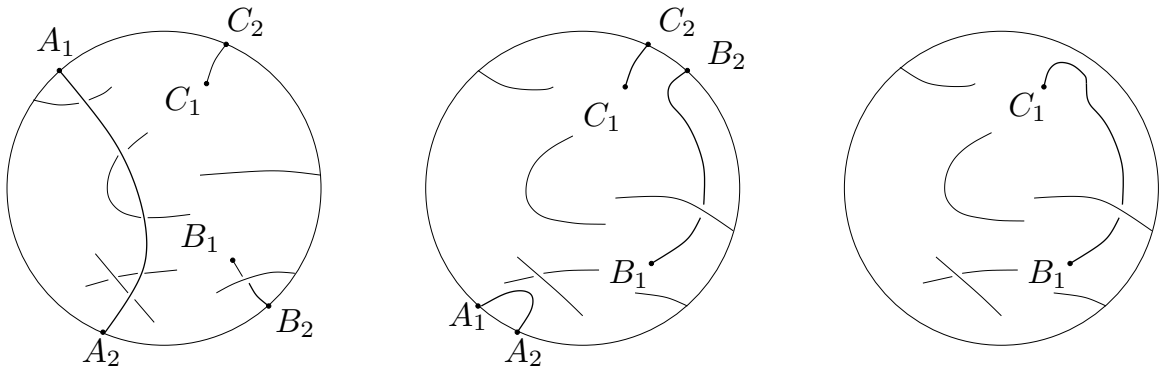


FIGURE 1.4. Removing upper arcs.

Using Type V Reidemeister moves isotope the diagram to move the point A_1 close enough to A_2 so that there is no arc with end points between A_1 and A_2 . In the process B_2 is moved closer to C_2 . Again use Type V moves to move B_2 even closer to C_2 if necessary so that there is no arc with end points between B_2 and C_2 . Use Type II and Type III moves to isotope the arc A_1A_2 so that there is no arc in the bigon

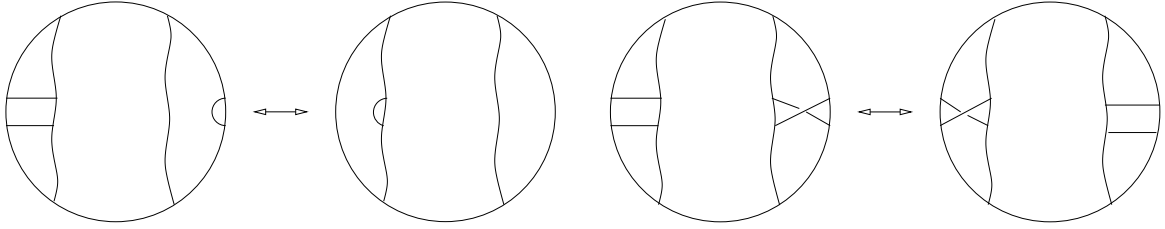


FIGURE 1.5. Type IV and Type V Reidemeister moves.

bounded by the arc A_1A_2 and the corresponding part of the boundary circle. Finally a Type IV move can be applied to get rid of the arc A_1A_2 . \square

REMARK 1.4.5. In the proof above we used an idea of Mroczkowski [Mro03a].

From now on we suppose that such isotopies have been performed.

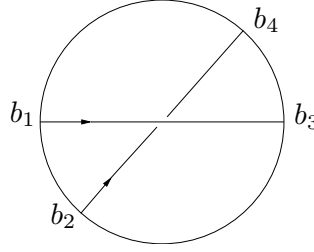
PROPOSITION 1.4.6. *If the diagram contains more than one crossing then a relation at a crossing can be deduced from the remaining relations. As a consequence if there is no affine unknot component then in the presentation of Theorem 1.4.1 one may choose to omit one Wirtinger-type relation so that the number of generators is one more than the number of relations. In other words it is a presentation with deficiency one.*

PROOF. If there are more than one crossings then the product of Wirtinger relations at the crossings, written in a certain order, is 1. Thus one relation can be deduced from the rest.

In the remaining case of fewer than two crossings there are the affine unknot, the non-affine unknot and the link with two components and one crossing (1_1^2 in Drobo-tukhina's table [Dro94], see Figure 1.6). Its fundamental group has a presentation

$$\langle b_1, b_2, b_3, b_4, c/b_2b_1 = b_3b_4, b_3 = c^{-1}b_1c, b_4 = c^{-1}b_1^{-1}b_2b_1c, b_1^{-1}b_2^{-1} = c^2, b_1 = b_3 \rangle.$$

The relation $b_2b_1 = b_3b_4$ is a consequence of the rest, so it can be dropped. \square

FIGURE 1.6. The link 1_1^2 .

1.4.3. The first homology group. From this presentation of the fundamental group we can deduce the first homology group.

COROLLARY 1.4.7. *Let L be a link with v components. If there exists one component of L whose number of intersection points with the canonical $\mathbb{R}P^2$ is odd then this component represents the non-trivial homology class of $\mathbb{R}P^3$, and $H_1(\mathbb{R}P^3 \setminus L) \cong \mathbb{Z}^v$. In the other case, L represents the trivial homology class of $\mathbb{R}P^3$ and $H_1(\mathbb{R}P^3 \setminus L) \cong \mathbb{Z}^v \oplus \mathbb{Z}_2$.*

PROOF. We know that the homology group $H = H_1(\mathbb{R}P^3 \setminus L)$ is the abelianization of the fundamental group $\pi_1(\mathbb{R}P^3 \setminus L)$. As a result of the abelianization, the Wirtinger-type relations and the relation

$$b_{p+i} = c^{-1} b_1^{\epsilon_1} b_2^{\epsilon_2} \cdots b_{i-1}^{\epsilon_{i-1}} b_i b_{i-1}^{-\epsilon_{i-1}} b_{i-2}^{-\epsilon_{i-2}} \cdots b_1^{-\epsilon_1} c, 1 \leq i \leq p$$

would identify all the b_i and a_j corresponding to the same k th component of L as an element $t_k \in H$, and also identify b_i and b_{p+i} . Thus

$$H = \langle c, t_1, t_2, \dots, t_v / ct_i = t_i c, t_i t_j = t_j t_i, \prod_{i=1}^v t_i^{\delta_i} = c^2 \rangle,$$

where δ_i is the sum of all ϵ_k , $0 \leq k \leq p$, such that b_k corresponds to the i th component, $1 \leq i \leq v$.

There are two cases:

Case 1: All δ_i are even. Write $\delta_i = 2k_i$, $k_i \in \mathbb{Z}$, $1 \leq i \leq v$. In this case $t_1^{2k_1} t_2^{2k_2} \cdots t_v^{2k_v} = c^2$, so $(ct_1^{-k_1} t_2^{-k_2} \cdots t_v^{-k_v})^2 = 1$. Let $u = ct_1^{-k_1} t_2^{-k_2} \cdots t_v^{-k_v}$. Then $u^2 = 1$ and $c = ut_1^{k_1} t_2^{k_2} \cdots t_v^{k_v}$, so

$$H = \langle t_1, t_2, \dots, t_v, u/t_i u = ut_i, t_i t_j = t_j t_i, u^2 = 1 \rangle \cong \mathbb{Z}^v \oplus \mathbb{Z}_2.$$

Case 2: There is a δ_i that is odd. Let $I = \{i, 1 \leq i \leq v : \delta_i = 2k_i + 1\}$ and $J = \{i, 1 \leq i \leq v\} \setminus I$. Let $i_0 = \min\{i, i \in I\}$. Then $\prod_{i \in I} t_i^{2k_i+1} \prod_{j \in J} t_j^{2k_j} = c^2$, so $\prod_{i \in I} t_i = c^2 (\prod_{i \in I} t_i^{-k_i})^2 (\prod_{j \in J} t_j^{-k_j})^2 = (c \prod_{1 \leq i \leq v} t_i^{-k_i})^2$. Let $u = c \prod_{1 \leq i \leq v} t_i^{-k_i}$. Then $\prod_{i \in I} t_i = u^2$. Since $i_0 \in I$ we have $t_{i_0} \prod_{i \in I \setminus \{i_0\}} t_i = u^2$, which implies that $t_{i_0} = u^2 \prod_{i \in I \setminus \{i_0\}} t_i^{-1}$. Also $c = u \prod_{1 \leq i \leq v} t_i^{k_i} = t^{1+2k_{i_0}} \prod_{i \in I \setminus \{i_0\}} t_i^{k_i - k_{i_0}} \prod_{j \in J} t_j^{k_j}$. So

$$H = \langle t_1, t_2, \dots, \hat{t}_{i_0}, \dots, t_v, u/t_i u = ut_i, t_i t_j = t_j t_i \rangle \cong \mathbb{Z}^v.$$

Note that in both cases the presentations of H do not depend on the diagrams (since the numbers δ_i 's don't).

Consider any component K of L . According to Poincaré Duality with \mathbb{Z}_2 coefficients there is a non-degenerate bilinear form $H_1(\mathbb{RP}^3, \mathbb{Z}_2) \times H_2(\mathbb{RP}^3, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$. More specifically, $\langle [K], [\mathbb{RP}^2] \rangle$ is exactly the mod 2 intersection number of the curve K and the surface \mathbb{RP}^2 (cf. Section 1.3 and Figure 1.2), which is $\delta_i \pmod{2}$. (In the case L is a knot this number is $(p \pmod{2}) = (\sum_{i=1}^p \epsilon_i \pmod{2})$.) When $\langle [K], [\mathbb{RP}^2] \rangle = 1$ we would have $[K]$ is non trivial in $H_1(\mathbb{RP}^3, \mathbb{Z}_2) \cong \mathbb{Z}_2$, and also $[\mathbb{RP}^2]$ is non trivial in $H_2(\mathbb{RP}^3, \mathbb{Z}_2) \cong \mathbb{Z}_2$. On the other hand when $\langle [K], [\mathbb{RP}^2] \rangle = 0$ we must have that $[K]$ is trivial in $H_1(\mathbb{RP}^3, \mathbb{Z}_2)$. \square

1.4.3.1. *Terminology.* We will call a link a *nontorsion link* if each of its component is null-homologous (each component of the link has an even number of intersection points with the canonical \mathbb{RP}^2 , in other words in its standard diagram the number of intersection points of each component with the boundary circle is a multiple of 4) The

first homology group of the complement of a nontorsion link is isomorphic to $\mathbb{Z}^v \oplus \mathbb{Z}_2$. The other links are called *torsion links*.

1.5. Twisted Alexander polynomial

The twisted Alexander polynomial in the following form was defined first by Turaev in [Tur02a] and was discussed further in [Tur02b, p. 27]. It receives attention recently in [HP04].

1.5.1. Twisted homomorphisms from $\mathbb{Z}[H]$ to $\mathbb{Z}[G]$.

1.5.1.1. *Definition.* Consider the complement of a link in \mathbb{RP}^3 . Fix a splitting of H as a product $H = G \times \text{Tors } H$ of the torsion part $\text{Tors } H = \langle u \rangle$ and a free part $G \cong H / \text{Tors } H$. Consider a representation (or a character) φ from $\text{Tors } H = \langle u \rangle$ to $\text{Aut}_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}^*$. If $|\text{Tors } H| = 1$ let $\varphi(u) = 1$; if $|\text{Tors } H| = 2$, let $\varphi(u) = -1$. It can be written as $\varphi(u) = (-1)^{|\text{Tors } H|+1}$. The map φ then induces a ring homomorphism from $\mathbb{Z}[H]$ to $\mathbb{Z}[G]$ by defining $\varphi(gu) = g\varphi(u)$. Corresponding to the case $|\text{Tors } H| = 1$, φ is the canonical projection from $\mathbb{Z}[H]$ to $\mathbb{Z}[G]$. Corresponding to the case $|\text{Tors } H| = 2$ it is a *twisted* homomorphism. The composition of φ and the canonical projection $pr : \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[H]$ gives us a ring homomorphism from $\mathbb{Z}[\pi]$ to $\mathbb{Z}[G]$.

1.5.1.2. *Notation.* From now on for simplicity of notation depending on the context we use the letter φ for the this map, either from $\mathbb{Z}[F]$ to $\mathbb{Z}[G]$, or from $\mathbb{Z}[\pi]$ to $\mathbb{Z}[G]$, or from $\mathbb{Z}[H]$ to $\mathbb{Z}[G]$.

1.5.1.3. *Dependence on splittings.* We discuss the dependence of φ on splittings of the first homology group. Suppose that $h \in H \subset \mathbb{Z}[H]$. Corresponding to a splitting $H = G \times \text{Tors } H$ there is a unique $g \in G$ and a unique $f \in \text{Tors } H$ such that $h = gf$ and so $\varphi(h) = g\varphi(f)$. Now let $H = G' \times \text{Tors } H$ be another splitting. Since $g \in H$, there is a unique $g' \in G'$ and a unique $f' \in \text{Tors } H$ such that $g = g'f'$. Then in this new splitting $h = g'(f'f)$. Define $\psi \in \text{Hom}(G, \text{Tors } H)$ by $\psi(g) = f'$. We can write

$h = g'(f'f) = g'\psi(g)f$, and $\varphi(h) = g'\varphi(\psi(g))\varphi(f)$. Using the isomorphism from G to G' identifying g and g' we see that under the new splitting $\varphi(h)$ is multiplied with $\varphi(\psi(g))$.

Note that in our case if the link is nontorsion then $|\text{Tors } H| = 2$ and splitting is not unique. Since $\varphi(\psi(G)) \subset \{-1, 1\}$, under φ any element $\sum_{h \in H} n_h h$ is mapped to $\sum_{h \in H} n_h \varphi(\psi(g))\varphi(h) = \sum_{h \in H} \pm n_h \varphi(h)$, thus the integral coefficients are only defined up to signs if we want to be independent from choices of splittings.

To avoid this problem from now on we will fix the splitting of H as in Corollary 1.4.7. In this splitting if a link is nontorsion then the free part of the first homology group is generated by the meridians. For example, for a nontorsion knot with the Wirtinger-type presentation of the fundamental group as in Theorem 1.4.1 we have $\varphi(a_i) = \varphi(b_j) = t, 1 \leq i \leq q, 1 \leq j \leq 2p$, and $\varphi(c) = \varphi(ut^k) = -t^k$, where $k = (\sum_{i=1}^p \epsilon_i)/2$. For a nontorsion link with v components, $\varphi(t_1^{m_1} t_2^{m_2} \dots t_v^{m_v} u^n) = t_1^{m_1} t_2^{m_2} \dots t_v^{m_v} (-1)^n$.

1.5.2. Twisted Alexander polynomial. Suppose L is a link in \mathbb{RP}^3 and let $\pi = \pi_1(\mathbb{RP}^3 \setminus L)$. Given a presentation $\pi = \langle x_1, \dots, x_n / r_1, \dots, r_m \rangle$ we construct an $m \times n$ matrix $[pr(\partial r_i / \partial x_j)]_{i,j}$, whose entries are elements of $\mathbb{Z}[H]$. It is called the Alexander–Fox matrix. Denote by $E(\pi)$ the ideal of $\mathbb{Z}[H]$ generated by the $(n-1) \times (n-1)$ -minors of the Alexander–Fox matrix. It is known that $E(\pi)$ does not depend on a presentation of π . Recall from Section 1.5.1 the twisted homomorphism $\varphi : \mathbb{Z}[H] \rightarrow \mathbb{Z}[G]$.

DEFINITION 1.5.1. The twisted Alexander polynomial of L is defined as

$$\Delta^\varphi(L) = \text{gcd } \varphi(E(\pi)) \in \mathbb{Z}[G] / \pm G.$$

Note that in a unique factorization domain the greatest common divisor is only defined up to units.

REMARK 1.5.2. a). If the link L is a torsion link then the twisted Alexander polynomial is exactly the usual (untwisted) Alexander polynomial $\Delta(L)$. If instead of the twisted map φ we use the canonical projection $\mathbb{Z}[H] \rightarrow \mathbb{Z}[G]$ (the torsion part of H is sent to 1) then we would also get the Alexander polynomial $\Delta(L)$.

b). Since the ring homomorphism $\varphi : \mathbb{Z}[H] \rightarrow \mathbb{Z}[G]$ is onto, the ideal $\varphi(E_1(\pi))$ is the same as the ideal generated by the $(n - 1) \times (n - 1)$ -minors of the matrix $[\varphi(\partial r_i / \partial x_j)]_{i,j}$, whose entries are in $\mathbb{Z}[G]$.

EXAMPLE 1.5.3 (The knot 2_1). Let K be the knot 2_1 in Drobotukhina's table, the only knot with two crossings. The fundamental group has a presentation

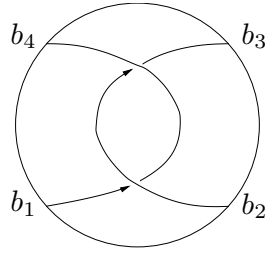


FIGURE 1.7. The knot 2_1 .

$$\pi = \langle b_1, b_2, b_3, b_4, c/b_2b_1 = b_4b_2 = b_3b_4, b_3 = c^{-1}b_1c, b_4 = c^{-1}b_1^{-1}b_2b_1c, b_1^{-1}b_2^{-1} = c^2 \rangle.$$

One Wirtinger-type relation can be deduced from the remaining relations, for example $b_4b_2 = b_3b_4$ can be deduced from the rest. Then the relation $b_2b_1 = b_4b_2$ is equivalent to $c = b_1c^3b_1$, and it follows that $\pi = \langle b_1, c/c = b_1c^3b_1 \rangle$, the only relator is $r = c^{-1}b_1c^3b_1$. The homology group is $H = \langle t, c/(ct)^2 = 1, ct = tc \rangle$, where t is the projection of the meridian b_1 . Let $u = ct$, so that $c = ut^{-1}$. Then $H = \langle u, t/u^2 = 1, tu = ut \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$. The twisted homomorphism associated with the above splitting of H is $\varphi : \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[t^{\pm 1}]$, determined by $\varphi(b_1) = t$ and $\varphi(c) = \varphi(u)t^{-1} = -t^{-1}$. We have $\partial r / \partial b_1 = -1 - b_1c^3$ and $\partial r / \partial c = 1 - b_1(1 + c + c^2)$.

So $\Delta_K^\varphi(t) = \gcd\{\varphi(\partial r/\partial b_1), \varphi(\partial r/\partial c)\} = \gcd\{-t^{-2}(t^2 - 1), -t^{-1}(t - 1)^2\} = t - 1$. On the other hand $\Delta_K(t) = \gcd\{-t^{-2}(t^2 + 1), -t^{-1}(t^2 + 1)\} = t^2 + 1$.

EXAMPLE 1.5.4 (Affine links). Suppose that L is an affine link, i.e. L can be isotoped so that it is contained inside a 3-ball in $\mathbb{R}P^3$, and so it is a nontorsion link. According to Theorem 1.4.1 and Proposition 1.4.6, the fundamental group of its complement is generated by a_1, a_2, \dots, a_q, c , where q is the number of crossings; together with $q - 1$ Wirtinger relations r_j involving a_i , and the relation $c^2 = 1$. Its Alexander–Fox matrix is the $q \times (q + 1)$ -matrix:

$$\begin{bmatrix} pr\left(\frac{\partial r_1}{\partial a_1}\right) & \dots & pr\left(\frac{\partial r_1}{\partial a_q}\right) & 0 \\ \vdots & & \vdots & \vdots \\ pr\left(\frac{\partial r_i}{\partial a_1}\right) & \dots & pr\left(\frac{\partial r_i}{\partial a_q}\right) & 0 \\ \vdots & & \vdots & \vdots \\ pr\left(\frac{\partial r_{q-1}}{\partial a_1}\right) & \dots & pr\left(\frac{\partial r_{q-1}}{\partial a_q}\right) & 0 \\ 0 & \dots & 0 & pr\left(\frac{\partial c^2}{\partial c}\right) \end{bmatrix},$$

where $\frac{\partial c^2}{\partial c} = 1 + c$.

Note that the $(q-1) \times q$ matrix $[pr(\partial r_i/\partial a_j)]_{i,j}$ is exactly the Alexander–Fox matrix of L viewed as a link in S^3 . It is immediate that the twisted Alexander polynomial of L is equal to $\varphi(1 + c) = 1 - 1 = 0$ multiplied with the Alexander polynomial of L viewed as a link in S^3 , thus $\Delta^\varphi(L) = 0$.

On the other hand $\Delta(L)$, the Alexander polynomial of L viewed as a link in $\mathbb{R}P^3$, will be twice the Alexander polynomial of L viewed as a link in S^3 , because in this situation $1 + c$ will be mapped to 2. This supports the result that the value of the Alexander polynomial of a knot complement evaluated at 1 is exactly the cardinality of the torsion part of the homology group (see [Tur86, p. 133], [Nic03, p. 69]).

1.6. Twisted Alexander polynomial and Reidemeister torsion

1.6.1. Reidemeister torsion of link complements. Let L be a link in \mathbb{RP}^3 .

The complement of L is $X = \mathbb{RP}^3 \setminus N(L)$, where $N(L)$ is an open tubular neighborhood of L , a collection of open solid tori. In term of the Euler characteristic, note that $0 = \chi(\mathbb{RP}^3) = \chi(X \cup \overline{N(L)}) = \chi(X) + \chi(\overline{N(L)}) - \chi(X \cap \overline{N(L)})$. Since $X \cap \overline{N(L)}$ is a collection of tori, $\chi(X \cap \overline{N(L)}) = 0$, and since $\chi(\overline{N(L)}) = 0$, it follows that $\chi(X) = 0$. The complement X is simple homotopic to a 2-dimensional cell complex Y which has one 0-cell σ^0 ; n 1-cells $\sigma_1^1, \dots, \sigma_n^1$; and m 2-cells $\sigma_1^2, \dots, \sigma_m^2$. Since X has zero Euler characteristic it follows that $m = n - 1$. The boundary maps are $\partial_1 = 0$ and $\partial_2(\sigma_i^2) = r_i$, where r_i is a word in σ_j^1 , giving a presentation of the fundamental group as $\pi = \pi_1(X) = \langle x_1, x_2, \dots, x_n/r_1, r_2, \dots, r_m \rangle$. This presentation is not necessarily the same as the presentation in Theorem 1.4.1, however.

Let \tilde{Y} be the maximal abelian cover of Y . Consider the cellular complexes of \tilde{Y} as modules over $\mathbb{Z}[H]$. Thus $C_0(\tilde{Y})$ has a basis $\{\tilde{\sigma}^0\}$, $C_1(\tilde{Y})$ has a basis $\{\tilde{\sigma}_1^1, \dots, \tilde{\sigma}_n^1\}$, and $C_2(\tilde{Y})$ has a basis $\{\tilde{\sigma}_1^2, \dots, \tilde{\sigma}_{n-1}^2\}$, where the tilde sign denotes a lift of the cell to \tilde{Y} . We have a chain complex of $\mathbb{Z}[H]$ -modules $C_2(\tilde{Y}) \xrightarrow{\partial_2} C_1(\tilde{Y}) \xrightarrow{\partial_1} C_0(\tilde{Y}) \rightarrow 0$. The boundary maps are obtained using Fox's Free Differential Calculus: $\partial_1(\tilde{\sigma}_i^1) = pr(x_i - 1)\tilde{\sigma}^0$ and $\partial_2(\tilde{\sigma}_i^2) = \sum_{j=1}^n pr(\frac{\partial r_i}{\partial x_j})\tilde{\sigma}_j^1$. As homomorphisms between modules, these maps can be represented by the matrices $[\partial_1]_i = pr(x_i) - 1, 1 \leq i \leq n$, and $[\partial_2]_{i,j} = pr(\frac{\partial r_i}{\partial x_j})$, $1 \leq i \leq n, 1 \leq j \leq n - 1$.

Denote the quotient field $Q(\mathbb{Z}[G])$ of $\mathbb{Z}[G]$ by $\mathbb{Q}(G)$. Using the homomorphism $\varphi : \mathbb{Z}[H] \rightarrow \mathbb{Z}[G] \hookrightarrow \mathbb{Q}(G)$, construct the tensor $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i(\tilde{Y})$, considered as a vector space over $\mathbb{Q}(G)$. We have a chain complex of vector spaces over $\mathbb{Q}(G)$:

$$C = (\mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_2(\tilde{Y}) \xrightarrow{\partial_2} \mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_1(\tilde{Y}) \xrightarrow{\partial_1} \mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_0(\tilde{Y}) \rightarrow 0).$$

The boundary maps are $[\partial_1]_i = \varphi(x_i) - 1$, and $[\partial_2]_{i,j} = \varphi(\frac{\partial r_j}{\partial x_i})$, $1 \leq i \leq n$, $1 \leq j \leq n-1$.

Denote by A the $(n-1) \times n$ matrix $[\partial_2]^t$. This matrix is obtained from the Alexander–Fox matrix by applying φ to its entries, resulting in entries belonging to $\mathbb{Z}[G]$.

First we exploit the fact that C is a chain. The condition $\partial_1 \circ \partial_2 = 0$ means for every $1 \leq i \leq n-1$ we must have:

$$\begin{aligned} 0 = \partial_1(\partial_2(\tilde{\sigma}_i^2)) &= \partial_1\left(\sum_{j=1}^n \varphi\left(\frac{\partial r_i}{\partial x_j}\right)\tilde{\sigma}_j^1\right) = \sum_{j=1}^n \varphi\left(\frac{\partial r_i}{\partial x_j}\right)\partial_1(\tilde{\sigma}_j^1) \\ &= \sum_{j=1}^n \varphi\left(\frac{\partial r_i}{\partial x_j}\right)(\varphi(x_j) - 1)\tilde{\sigma}^0. \end{aligned}$$

Thus for every $1 \leq i \leq n-1$, $\sum_{j=1}^n \varphi\left(\frac{\partial r_i}{\partial x_j}\right)(\varphi(x_j) - 1) = 0$.

Denote the columns of the matrix A by u_i , $1 \leq i \leq n$, and denote the $(n-1) \times (n-1)$ matrix obtained from A by omitting the column u_i by A_i . The condition above means that $\sum_{j=1}^n (\varphi(x_j) - 1)u_j = 0$.

For any $i > j$ we have (a hat over a column denotes that the column is omitted)

$$\begin{aligned} (\varphi(x_j) - 1) \det A_i &= \det[u_1, \dots, u_{j-1}, (\varphi(x_j) - 1)u_j, u_{j+1}, \dots, \hat{u}_i, \dots, u_n] \\ &= \det[u_1, \dots, u_{j-1}, -\sum_{k \neq j} (\varphi(x_k) - 1)u_k, u_{j+1}, \dots, \hat{u}_i, \dots, u_n] \\ &= \det[u_1, \dots, u_{j-1}, -(\varphi(x_i) - 1)u_i, u_{j+1}, \dots, \hat{u}_i, \dots, u_n] \\ &= (-1)^{i-j+1}(\varphi(x_i) - 1) \det A_j. \end{aligned}$$

Thus we have shown that for any i and j ,

$$(1.6.1) \quad (\varphi(x_i) - 1) \det A_j = \pm(\varphi(x_j) - 1) \det A_i.$$

Because H has at least one free generator (see Corollary 1.4.7), the image $\varphi(\pi)$ cannot be $\{1\}$. Thus there is at least one generator x_i such that $\varphi(x_i) \neq 1$. The property $\partial_1(\tilde{\sigma}_i^1) = (\varphi(x_i) - 1)\tilde{\sigma}^0$ implies $\partial_1(\frac{1}{\varphi(x_i)-1}\tilde{\sigma}_i^1) = \tilde{\sigma}^0$, so ∂_1 is onto. Therefore

the chain C is exact if and only if ∂_2 is injective, which means the rank of its matrix is exactly $n - 1$. Thus the chain C is acyclic if and only if the matrix A has a nonzero $(n - 1) \times (n - 1)$ minor.

The Reidemeister torsion of the chain C with respect to the map φ is the Reidemeister torsion $\tau^\varphi(Y)$ of Y , and since torsion is a simple homotopy invariant, it is also the torsion $\tau^\varphi(X)$ of X . If C is not acyclic then its Reidemeister torsion is defined to be 0. For a moment, assume that C is acyclic. Then the torsion is given by the formula $\tau^\varphi(X) = \tau(C) = [(\partial_2 c_2) \tilde{c}_0 / c_1]$. In this formula c_i is a basis for the $\mathbb{Q}(G)$ -vector space $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_i(\tilde{Y})$, \tilde{c}_0 is a lift of the basis c_0 of $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_0(\tilde{Y})$ to $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_1(\tilde{Y})$, $(\partial_2 c_2) \tilde{c}_0$ denotes a new basis of $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_1(\tilde{Y})$ obtained by combining the vectors in $\partial_2 c_2$ and \tilde{c}_0 together in that order, finally $[(\partial_2 c_2) \tilde{c}_0 / c_1]$ is the determinant of the change of base matrix. As an element of $\mathbb{Q}(G) / \pm G$ the torsion does not depend on the choice of bases or the choice of lifts.

Take the standard bases of $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_i(\tilde{Y})$ given by $\tilde{\sigma}_j^i$ as above. A lift of $c_0 = \{\tilde{\sigma}_0\}$ is $\{\frac{1}{\varphi(x_i)-1} \tilde{\sigma}_i^1\}$, for any i . Then

$$\begin{aligned} \tau^\varphi(X) &= \left[\left(\sum_{j=1}^n \varphi\left(\frac{\partial r_1}{\partial x_j}\right) \tilde{\sigma}_j^1, \dots, \sum_{j=1}^n \varphi\left(\frac{\partial r_{n-1}}{\partial x_j}\right) \tilde{\sigma}_j^1, \frac{1}{\varphi(x_i)-1} \tilde{\sigma}_i^1 \right) / (\tilde{\sigma}_1^1, \dots, \tilde{\sigma}_n^1) \right] \\ &= \det \begin{bmatrix} \varphi\left(\frac{\partial r_1}{\partial x_1}\right) & \dots & \varphi\left(\frac{\partial r_{n-1}}{\partial x_1}\right) & 0 \\ \vdots & & \vdots & \vdots \\ \varphi\left(\frac{\partial r_1}{\partial x_{i-1}}\right) & \dots & \varphi\left(\frac{\partial r_{n-1}}{\partial x_{i-1}}\right) & 0 \\ \varphi\left(\frac{\partial r_1}{\partial x_i}\right) & \dots & \varphi\left(\frac{\partial r_{n-1}}{\partial x_i}\right) & \frac{1}{\varphi(x_i)-1} \\ \varphi\left(\frac{\partial r_1}{\partial x_{i+1}}\right) & \dots & \varphi\left(\frac{\partial r_{n-1}}{\partial x_{i+1}}\right) & 0 \\ \vdots & & \vdots & \vdots \\ \varphi\left(\frac{\partial r_1}{\partial x_n}\right) & \dots & \varphi\left(\frac{\partial r_{n-1}}{\partial x_n}\right) & 0 \end{bmatrix} \\ &= \frac{(-1)^{i+n}}{\varphi(x_i)-1} \det A_i. \end{aligned}$$

Thus if $\varphi(x_i) \neq 1$ then $\tau^\varphi(X) = \pm \det A_i / (\varphi(x_i) - 1)$. Note that in view of Formula (1.6.1) if $\varphi(x_j) = 1$ then $\det(A_j) = 0$. In view of the acyclicity condition of C the following formula is correct for all i , whether the chain C is acyclic or not:

$$(1.6.2) \quad (\varphi(x_i) - 1)\tau^\varphi(X) = \pm \det A_i \in \mathbb{Q}(G) / \pm G.$$

THEOREM 1.6.1. *The Reidemeister torsion and the twisted Alexander polynomial of the complement of a nontorsion link are the same.*

PROOF. According to Definition 1.5.1 and Formula (1.6.2), we have $\Delta^\varphi(X) = \gcd\{\det A_1, \dots, \det A_n\} = \gcd\{(\varphi(x_1) - 1)\tau^\varphi(X), \dots, (\varphi(x_n) - 1)\tau^\varphi(X)\}$. First we prove the following assertion.

$$\text{ASSERTION 1.6.2. } \gcd\{\varphi(x_1) - 1, \varphi(x_2) - 1, \dots, \varphi(x_n) - 1\} = 1 \in \mathbb{Z}[G] / \pm G.$$

PROOF. *Case 1: L has one component (a knot).* In this case $H = \langle t, u/tu = ut, u^2 = 1 \rangle$, $pr(x_i) = t^{m_i}u^{n_i}$, and the twisted map is $\varphi(x_i) = t^{m_i}(-1)^{n_i}$. Let $d = \gcd\{\varphi(x_1) - 1, \varphi(x_2) - 1, \dots, \varphi(x_n) - 1\} \in \mathbb{Z}[G]$. The following two identities :

$$(t^{m_i}(-1)^{n_i} - 1) + t^{m_i}(-1)^{n_i}(t^{m_j}(-1)^{n_j} - 1) = t^{m_i+m_j}(-1)^{n_i+n_j} - 1,$$

$$(t^{m_i}(-1)^{n_i} - 1) - t^{m_i-m_j}(-1)^{n_i-n_j}(t^{m_j}(-1)^{n_j} - 1) = t^{m_i-m_j}(-1)^{n_i-n_j} - 1,$$

(compare [Lic97, p. 117]) imply that $d \mid (t^{\sum_{i=1}^n \alpha_i m_i} (-1)^{\sum_{i=1}^n \alpha_i n_i} - 1)$ for any $\alpha_i \in \mathbb{Z}$.

Since $t \in pr(\pi)$, there are $\alpha_i \in \mathbb{Z}$ such that $t = \prod_{i=1}^n pr(x_i^{\alpha_i}) = t^{\sum_{i=1}^n \alpha_i m_i} u^{\sum_{i=1}^n \alpha_i}$, which implies $\sum_{i=1}^n \alpha_i m_i = 1$ and $\sum_{i=1}^n \alpha_i$ is even. Thus $d \mid (t - 1)$, hence either $d = 1$ or $d = t - 1$, up to $\pm t^k, k \in \mathbb{Z}$. Since $u \in pr(\pi)$ there is at least an i_0 such that n_{i_0} is odd, so that $\varphi(x_{i_0}) - 1 = -t^{m_{i_0}} - 1$. Note that 1 is not a zero of $-t^{m_{i_0}} - 1$, so $t - 1$ is not a factor of $-t^{m_{i_0}} - 1$, hence $\gcd\{t - 1, -t^{m_{i_0}} - 1\} = 1$, which gives us the conclusion that $d = 1$.

Case 2: L has at least two components. Let $v \geq 2$ be the number of components. In this case $pr(x_i) = t_1^{m_1^i} t_2^{m_2^i} \cdots t_v^{m_v^i} u^{n_i}$ and $\varphi(x_i) = t_1^{m_1^i} t_2^{m_2^i} \cdots t_v^{m_v^i} (-1)^{n_i}$. Let $t_2 = t_3 = \cdots = t_v = 1$ then by applying the above argument for knots to t_1 we have $\gcd\{\varphi(x_1) - 1, \varphi(x_2) - 1, \dots, \varphi(x_n) - 1\} = 1$, so this would also be true in general. \square

Returning to the proof of the theorem, now it follows immediately from Assertion 1.6.2 that $\tau^\varphi(X) \in \mathbb{Z}[G]$ and $\gcd\{(\varphi(x_1) - 1)\tau^\varphi(X), \dots, (\varphi(x_n) - 1)\tau^\varphi(X)\} = \tau^\varphi(X)$. The proof is completed. \square

THEOREM 1.6.3. *If L is a torsion knot and t is the generator of the first homology group of the complement then $\tau_L^\varphi(t) = \Delta_L^\varphi(t)/(t - 1) \in \mathbb{Z}[t^{\pm 1}, (t - 1)^{-1}]$. If L is a torsion link with a least two components then the Reidemeister torsion and the twisted Alexander polynomial of its complement are the same.*

PROOF. The proof follows the same lines as the proof of Theorem 1.6.1. In this case $|\text{Tors } H_1(X)| = 1$.

Case 1: L has one component (a knot). In this case $H = \langle t \rangle$ and the twisted map is given by $\varphi(x_i) = t^{m_i}$. By a similar argument to Assertion 1.6.2, using the two identities:

$$t^{m_i} + t^{m_i}(t^{m_j} - 1) = t^{m_i+m_j} - 1,$$

$$(t^{m_i} - 1) - t^{m_i-m_j}(t^{m_j} - 1) = t^{m_i-m_j} - 1,$$

we have $\gcd\{\varphi(x_i) - 1, 1 \leq i \leq n\} = (t - 1)$. Suppose that $\tau^\varphi(X) = f/g$, where $f, g \in \mathbb{Z}[t^{\pm 1}]$ and $\gcd(f, g) = 1$. Then $\Delta^\varphi(X) = \gcd\{(\varphi(x_i) - 1)f/g, 1 \leq i \leq n\}$, so $g\Delta^\varphi(X) = \gcd\{(\varphi(x_i) - 1)f, 1 \leq i \leq n\} = f \gcd\{\varphi(x_i) - 1, 1 \leq i \leq n\} = f(t - 1)$. Hence $\Delta^\varphi(X) = (t - 1)f/g = (t - 1)\tau^\varphi(X)$.

Case 2: L has at least two components. In this case H is generated by t_1, t_2, \dots, t_v where $v \geq 2$, and the twisted map is $\varphi(x_i) = t_1^{m_1^i} t_2^{m_2^i} \cdots t_v^{m_v^i}$. By subsequently letting $t_j = 1$ for all $j \neq i$ and applying the above argument for knots to t_i we obtain

$\gcd\{\varphi(x_1) - 1, \varphi(x_2) - 1, \dots, \varphi(x_n) - 1\} = \gcd\{t_1 - 1, t_2 - 1, \dots, t_v - 1\} = 1$. Hence $\Delta^\varphi(X) = \tau^\varphi(X)$. \square

REMARK 1.6.4. Recall (see Remark 1.5.2) that for a torsion link the twisted Alexander polynomial becomes the Alexander polynomial. With a virtually identical proof, the statement of Theorem 1.6.3 is true for all links if we replace the twisted map φ by the canonical projection $\mathbb{Z}[H] \rightarrow \mathbb{Z}[G]$, and replace the twisted Alexander polynomial by the Alexander polynomial.

1.6.2. Comparison with other twisted Alexander polynomials. Among the first people who studied twisted Alexander polynomials were Lin [**Lin01**], Wada [**Wad94**], Kitano [**Kit96**], Kirk-Livingston [**KL99**]. Except for Lin's construction of his polynomial for knots based on Seifert surfaces, other constructions are based on that of Wada. We briefly outline Wada's construction (not in full generality) to show its relationship with our polynomial.

Suppose $\pi = \langle x_1, \dots, x_m/r_1, \dots, r_{m-1} \rangle$ is a presentation of deficiency one of the group $\pi_1(X)$. Let $\alpha : \pi \rightarrow G \cong \langle t_1, t_2, \dots, t_v/t_i t_j = t_j t_i \rangle \cong \mathbb{Z}^v$ be a surjective group homomorphism. Let $\rho : \pi \rightarrow \mathrm{GL}(n, \mathbb{C})$ be a representation of π . Define a ring homomorphism $\phi : \mathbb{Z}[\pi] \rightarrow M(n, \mathbb{C}[G])$ by letting $\phi(x) = \alpha(x)\rho(x)$ for $x \in \pi$ then extend linearly (or equivalently one may first extend α linearly to a ring homomorphism $\tilde{\alpha} : \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[G]$ and extend ρ linearly to a ring homomorphism $\tilde{\rho} : \mathbb{Z}[\pi] \rightarrow M(n, \mathbb{C})$, then write $\phi = \tilde{\alpha} \otimes \tilde{\rho}$).

Consider the $(m-1) \times m$ matrix M whose the (i, j) entry is $\phi(\partial r_i / \partial x_j) \in M(n, \mathbb{C}[G])$. Let M_j be the $(m-1) \times (m-1)$ matrix obtained from M by removing the j th column. View M_j as an $n(m-1) \times n(m-1)$ matrix whose entries are in $\mathbb{C}[G]$. Supposing that $\phi(x_j) \neq I$, we define the twisted Alexander polynomial as

$$\Delta^\rho(X) = \frac{\det M_j}{\det \phi(1 - x_j)} \in \mathbb{C}(G) = Q(\mathbb{C}[G]).$$

Wada proved that this polynomial is independent of the choice of j and the choice of a presentation of π , and is defined up to a factor in $\pm G$.

Let us compare Wada's polynomial with Turaev's one. Fix a splitting $H = G \times \text{Tors } H$. Suppose that $\varphi \in \text{Hom}(\text{Tors } H, \mathbb{C}^*)$ is given. Let α be as above, and ρ be the composition of the maps $\pi \rightarrow H \xrightarrow{\beta} \text{Tors } H \xrightarrow{\varphi} \{\pm 1\} \subset \text{GL}(1, \mathbb{C})$; here the first arrow is the canonical projection map, and β maps an element $gh \in H$ where $g \in G$ and $h \in \text{Tors } H$ to h . Then $\phi = \tilde{\alpha} \otimes \tilde{\rho}$ is exactly the twisted map in Section 1.5.1. Thus $\Delta^\rho(X)$ here is exactly the torsion $\tau^\varphi(X)$, in view of Formula (1.6.2), and the relationships of this polynomial with Turaev's polynomial are provided in Theorems 1.6.1 and 1.6.3.

REMARK 1.6.5. Milnor proved in [Mil62] the identification between Alexander polynomial and Reidemeister torsion for knot complements in S^3 . Kitano [Kit96] proved the identification between Wada's twisted Alexander polynomial and Reidemeister torsion, also for knot complements in S^3 . Kirk–Livingston [KL99] generalized this result to general CW-complex, but considered only a one variable twisted Alexander polynomial associated with an infinite cyclic cover of the complex. Turaev has also studied this problem, see [Tur02b, p. 28]. The proof above of Theorem 1.6.1 is close to Milnor's original proof and has the advantage of being straight forward and elementary.

1.7. A skein relation for the twisted Alexander polynomial

1.7.1. The one variable twisted Alexander polynomial.

1.7.1.1. *Definition.* When L is a link of v components, the twisted Alexander polynomial of its complement X is a polynomial in v variables t_1, t_2, \dots, t_v . The one variable twisted Alexander polynomial is obtained by identifying all $t_i, 1 \leq i \leq v$ as a single variable t . Thus the one variable polynomial $\Delta^{\varphi'}(X)$ is obtained from $\Delta^\varphi(X)$ by replacing φ by φ' , where φ' is the composition of φ with the canonical projection

from $\mathbb{Z}[t_1^{\pm 1}, \dots, t_v^{\pm 1}]$ to $\mathbb{Z}[t^{\pm 1}]$ (recall Section 1.5.1). We write $\mathbb{Q}(t) = Q(\mathbb{Z}[t, t^{-1}])$, the quotient field of the integral domain of Laurent polynomials with integer coefficients. We also write $\Delta^{\varphi'}(X)$ as $\Delta_L^{\varphi'}(t)$ and $\tau^{\varphi'}(X)$ as $\tau_L^{\varphi'}(t)$.

1.7.1.2. *Relations with Reidemeister torsion.*

THEOREM 1.7.1. *If L is a nontorsion link then the Reidemeister torsion $\tau_L^{\varphi'}(t)$ and the one variable twisted Alexander polynomial $\Delta_L^{\varphi'}(t)$ are the same.*

THEOREM 1.7.2. *If L is a torsion link then the Reidemeister torsion and the one variable twisted Alexander polynomial are related by the formula $\tau_L^{\varphi'}(t) = \Delta_L^{\varphi'}(t)/(t - 1) \in \mathbb{Z}[t^{\pm 1}, (t - 1)^{-1}]$.*

The proofs of the two theorems are identical to the proofs for the cases of knots of Theorems 1.6.1 and 1.6.3.

As a consequence of general symmetry property of Reidemeister torsion, we have:

THEOREM 1.7.3. *The Reidemeister torsion $\tau_L^{\varphi'}(t)$ is symmetric, that is $\tau_L^{\varphi'}(t^{-1}) = \tau_L^{\varphi'}(t)$ up to $\pm t^n, n \in \mathbb{Z}$, as elements in $\mathbb{Q}(t)$.*

From this we derive the following:

THEOREM 1.7.4. *The one variable twisted Alexander polynomial is symmetric, that is $\Delta_L^{\varphi'}(t^{-1}) = \Delta_L^{\varphi'}(t)$ up to $\pm t^n, n \in \mathbb{Z}$, as elements in $\mathbb{Z}[t^{\pm 1}]$.*

PROOF. If L is a nontorsion link then according to Theorem 1.7.1, $\Delta_L^{\varphi'}(t^{-1}) = \tau_L^{\varphi'}(t^{-1}) = \tau_L^{\varphi'}(t) = \Delta_L^{\varphi'}(t)$. On the other hand if L is a torsion link then according to Theorem 1.7.2, $\Delta_L^{\varphi'}(t^{-1}) = (t^{-1} - 1)\tau_L^{\varphi'}(t^{-1}) = t^{-1}(1 - t)\tau_L^{\varphi'}(t) = \Delta_L^{\varphi'}(t)$, up to $\pm t^n, n \in \mathbb{Z}$. \square

1.7.2. A skein relation for torsion with indeterminacies. Let L be an oriented link. Consider a crossing of L . Let B be an open 3-ball that encloses this

crossing and intersects L at four points. Let $V = \mathbb{RP}^3 \setminus (B \cup N(L))$, where $N(L)$ is an open tubular neighborhood of L ; see Figure 1.8.

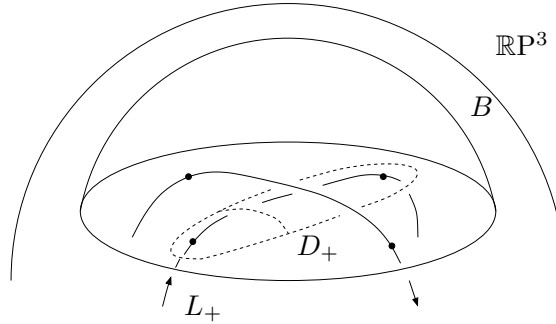


FIGURE 1.8. At the crossing.

Take a triangulation of V . There is a deformation retraction of the complement of L_α , $\alpha \in \{+, -, 0\}$, onto $X_\alpha = V \cup D_\alpha$, where D_α is a disk glued to ∂V along a simple loop ∂D_α circling two intersection points of B and L_α as in Figure 1.9 so that $V \cup D_\alpha$ has a cell decomposition consists of the cells of V plus the the disk D_α . We can assume that the loops ∂D_α have a common point.

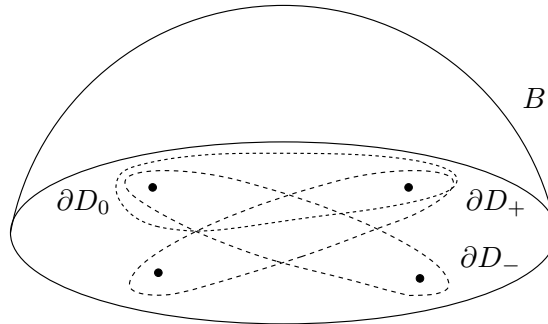


FIGURE 1.9. The curves ∂D_α .

1.7.2.1. *Smoothing of crossings and classes of links.* When the smoothing operation is done at a crossing L_0 may no longer be in the same torsion class with L_+ and L_- . The three links L_+ , L_- and L_0 are in the same torsion class in the following cases:

- i). there is one component of L_+ which is not involved at the crossing that is 1-homologous (cf. Section 1.4.3.1). In this case L_+ , L_- and L_0 are all torsion links.

ii). the two strands of L_+ at the crossing come from one component, and L_0 is a nontorsion link. In this case L_+ , L_- and L_0 are all nontorsion links (cf. Figure 1.11).

iii). the two strands of L_+ at the crossing come from two different components, and L_+ is nontorsion. In this case L_+ , L_- and L_0 are all nontorsion links (cf. Figure 1.13).

In what follows we will need the condition that the links L_α belong to the same torsion class, so that $\text{Tors } H_1(X_\alpha)$ are the same. Therefore throughout the rest of Section 1.7 we will assume that this condition is satisfied.

1.7.2.2. *The chain complexes C_α and C .* Fix an $\alpha \in \{+, -, 0\}$. Let \tilde{X}_α be the $D = \mathbb{Z} \times \text{Tors } H_1(X_\alpha)$ cover of X_α corresponding to the kernel of the map $proj_\alpha : \pi_1(X_\alpha) \rightarrow H_1(X_\alpha) \rightarrow G \times \text{Tors } H_1(X_\alpha) \rightarrow \{t^m : m \in \mathbb{Z}\} \times \text{Tors } H_1(X_\alpha)$. Let \tilde{V} be the inverse image of V under the covering map. The triangulation of V induces a CW-complex structure on \tilde{V} .

Under the condition that L_α are in the same torsion class \tilde{X}_α can be constructed in a different way as follows. Take \tilde{V} to be the cover of V corresponding to the kernel of the map $\pi_1(V) \rightarrow H_1(V) \rightarrow G \times \text{Tors } H_1(V) \rightarrow \{t^m : m \in \mathbb{Z}\} \times \text{Tors } H_1(V)$. Noting that $\text{Tors } H_1(V) = \text{Tors } H_1(X_\alpha)$ for $\alpha = +, -, 0$, we construct \tilde{X}_α from \tilde{V} by gluing $|\mathbb{Z} \times \text{Tors } H_1(X_\alpha)|$ copies of D_α along the lifts of $\partial D_\alpha \subset V$.

Consider the ring homomorphism $\varphi' : \mathbb{Z}[\{t^m : m \in \mathbb{Z}\} \times \text{Tors } H_1(X_\alpha)] \rightarrow \mathbb{Z}[t^{\pm 1}] \hookrightarrow \mathbb{Q}(t)$, which does not depend on α . Let $C_\alpha = \mathbb{Q}(t) \otimes_{\mathbb{Z}[\mathbb{Z} \times \text{Tors } H_1(X_\alpha)], \varphi'} C_*(\tilde{X}_\alpha, \mathbb{Z})$ and let $C = \mathbb{Q}(t) \otimes_{\mathbb{Z}[\mathbb{Z} \times \text{Tors } H_1(X_\alpha)], \varphi'} C_*(\tilde{V}, \mathbb{Z})$, both considered as chain complexes of $\mathbb{Q}(t)$ -vector spaces. Note that C does not depend on α .

1.7.2.3. *Relations among $\tau(C_\alpha)$.* Since $C_i(\tilde{V}, \mathbb{Z}) \hookrightarrow C_i(\tilde{X}_\alpha, \mathbb{Z})$ is an inclusion, the induced map $\mathbb{Q}(t) \otimes_{\varphi'} C_*(\tilde{V}, \mathbb{Z}) \hookrightarrow \mathbb{Q}(t) \otimes_{\varphi'} C_*(\tilde{X}_\alpha, \mathbb{Z})$ is injective, and we have the short exact sequence of chain complexes of $\mathbb{Q}(t)$ -vector spaces

$$(1.7.1) \quad 0 \rightarrow C \rightarrow C_\alpha \rightarrow C_\alpha/C \rightarrow 0.$$

Choose a fundamental family of cells (i.e. a family of lifted cells) for \tilde{V} , which provides a basis for the chain C . A fundamental family of cells of \tilde{X}_α is obtained from the family of \tilde{V} by adding a lift of D_α . We can choose these lifts D_α so that the loops $\partial\tilde{D}_\alpha$ have a common point in \tilde{V} , which is a lift of the common point of ∂D_α in V . Consider the chain C_α/C . Recalling that X_α is the result of gluing the disk D_α to V , one has that $C_\alpha/C = (0 \rightarrow \mathbb{Q}(t)[\tilde{D}_\alpha] \rightarrow 0 \rightarrow 0)$. Thus only the second homology group of C_α/C is non-trivial, and it is a one-dimensional vector space over $\mathbb{Q}(t)$ generated by \tilde{D}_α . The torsion of this chain complex with homology bases is $\tau(C_\alpha/C, h) = 1$ up to a sign.

Suppose that the chain complex C_α is acyclic. The product formula for torsion (1.2.6) applied to the short exact sequence of the pair (C_α, C) above (1.7.1) gives:

$$(1.7.2) \quad \tau(C_\alpha) = \pm\tau(C, h)\tau(C_\alpha/C, h)\tau(\mathcal{H}_\alpha) = \pm\tau(C, h)\tau(\mathcal{H}_\alpha),$$

where \mathcal{H}_α denotes the long exact homological sequence of the pair (C_α, C) , with a chosen basis:

$$\mathcal{H}_\alpha = (\cdots \rightarrow H_i(C) \rightarrow H_i(C_\alpha) \rightarrow H_i(C_\alpha/C) \rightarrow H_{i-1}(C) \rightarrow \cdots)$$

Since C_α is exact the sequence \mathcal{H}_α is reduced to

$$0 \rightarrow H_2(C_\alpha/C) \xrightarrow{\partial} H_1(C) \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0,$$

so $H_1(C) \cong H_2(C_\alpha/C) \cong \mathbb{Q}(t)$ and $\tau(\mathcal{H}_\alpha) = \det(\partial)$. Let y be the chosen basis of the one-dimensional $\mathbb{Q}(t)$ -vector space $H_1(C)$. Then $\partial[\tilde{D}_\alpha] = [\partial\tilde{D}_\alpha] = \gamma_\alpha y$ for some $\gamma_\alpha \in \mathbb{Q}(t)$.

Formula (1.7.2) now gives

$$(1.7.3) \quad \tau(C_\alpha) = \pm\gamma_\alpha\tau(C, h).$$

As can be seen from the proof the product formula in [Mil66, p. 160] or from the corresponding formula for sign-refined torsion (1.2.7), the sign \pm in (1.7.3) above depends only on the ranks of the vector spaces in the chains C_α , C and \mathcal{H}_α , thus does not depend on α (to the extent that C_α is assumed to be acyclic).

Under the assumption that there is at least one $\alpha_0 \in \{+, -, 0\}$ such that C_{α_0} is acyclic, we show that (1.7.3) above still holds when C_α is not acyclic. When C_α is not acyclic, by definition $\tau(C_\alpha) = 0$. We will show that γ_α is zero, i.e. the boundary map $\partial : H_2(C_\alpha/C) \rightarrow H_1(C)$ is zero, and so 1.7.3 holds for C_α . Suppose the contrary, $\gamma_\alpha \neq 0$. Because $H_1(C) \cong H_2(C_{\alpha_0}/C) \cong \mathbb{Q}(t) \cong H_2(C_\alpha/C)$, if ∂ is not zero it must be a bijection. The long exact sequence \mathcal{H}_α shows that $H_1(C_\alpha) = 0$. Note that $\text{rank}(\mathbb{Q}(t) \otimes_{\mathbb{Z}[\mathbb{Z} \times \text{Tors } H_1(\tilde{X}_\alpha), \varphi']} C_i(\tilde{X}_\alpha, \mathbb{Z}))$ is exactly the number of i -cells of X_α . This implies that $0 = \chi(X_\alpha) = \chi(C_\alpha) = \text{rank}(H_0(C_\alpha)) + \text{rank}(H_2(C_\alpha))$. Thus $H_0(C_\alpha) = H_1(C_\alpha) = H_2(C_\alpha) = 0$ i.e. C_α is acyclic, a contradiction.

1.7.2.4. *Relations among γ_α .* In view of (1.7.3) to further study relations among $\tau(C_\alpha)$ we now try to find a relation among γ_α .

Consider $\gamma_\alpha y = [\partial \tilde{D}_\alpha] \in H_1(C)$. Let a , b , c and d be oriented simple meridian loops with a common base point, circling the four intersection points between L and B as in Figure 1.10. Topologically the boundary of the disk D_α are: $\partial D_+ = bd$,

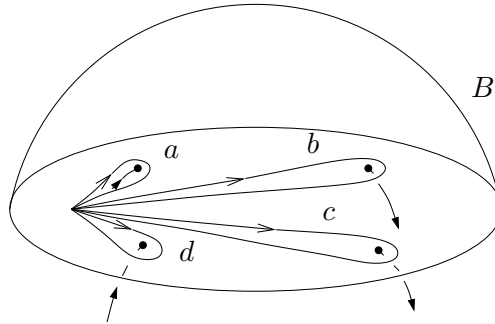


FIGURE 1.10. The loops a , b , c and d .

$\partial D_- = ac$, and $\partial D_0 = ab$. Thus it is necessary to find a relation between $[bd]$, $[ac]$ and

$[\tilde{a}\tilde{b}]$ as elements in the $\mathbb{Q}(t)$ -vector space $H_1(C)$. Recalling the map $proj_\alpha$ in Section 1.7.2.2, we have $proj_\alpha(a) = proj_\alpha(d) = t^{-1}$ and $proj_\alpha(b) = proj_\alpha(c) = t$.

The lift of the loop $\eta = (bd)(ab)(ca)(ba)^{-1} \subset V$ to $\tilde{V} \subset \tilde{X}_\alpha$ is

$$\tilde{\eta} = \tilde{b}\tilde{d} + proj_\alpha(bd)\tilde{a}\tilde{b} + proj_\alpha(bdab)\tilde{c}\tilde{a} - proj_\alpha(bdabca((ba)^{-1})^{-1})\tilde{b}\tilde{a} = \tilde{b}\tilde{d} + \tilde{a}\tilde{b} + \tilde{c}\tilde{a} - \tilde{b}\tilde{a}$$

(recalling that a common base point was chosen). Note that $ca = a^{-1}aca$, and the lift of $a^{-1}aca$ is the image of $\tilde{a}\tilde{c}$ under the action of the element of the deck transformation of \tilde{X}_α represented by a^{-1} . That element is $proj_\alpha(a^{-1}) = t$, thus $\tilde{c}\tilde{a} = proj_\alpha(a^{-1})\tilde{a}\tilde{c} = t\tilde{a}\tilde{c}$. Similarly, $ba = a^{-1}aba$, and so $\tilde{b}\tilde{a} = proj_\alpha(a^{-1})\tilde{a}\tilde{b} = t\tilde{a}\tilde{b}$. Thus

$$\tilde{\eta} = \tilde{b}\tilde{d} + \tilde{a}\tilde{b} + t\tilde{a}\tilde{c} - t\tilde{a}\tilde{b} = \tilde{b}\tilde{d} + (1-t)\tilde{a}\tilde{b} + t\tilde{a}\tilde{c}.$$

We can write $\eta = b(dabc)b^{-1}$. Since $dabc$ is contractible in V , η is contractible in V , so $\tilde{\eta}$ is a boundary in $C_1(\tilde{V}, \mathbb{Z})$. The corresponding element $[\tilde{\eta}]$ in $H_1(\mathbb{Q}(t) \otimes_{\varphi'} C_*(\tilde{V}, \mathbb{Z}))$ must be zero. Thus we obtain $\gamma_+y + (1-t)\gamma_0y + t\gamma_-y = 0$ in $H_1(C)$, hence $\gamma_+ + (1-t)\gamma_0 + t\gamma_- = 0$ in $\mathbb{Q}(t)$.

Formula (1.7.3) now gives us, under the assumption that there is at least one $\alpha_0 \in \{+, -, 0\}$ such that C_{α_0} is acyclic, the formula $\tau(C_+) + (1-t)\tau(C_0) + t\tau(C_-) = 0$. But this formula is also trivially correct when none of the C_α are acyclic, since in that case all three torsions are zero. Thus we obtain the following theorem:

THEOREM 1.7.5. *If L_+ , L_- and L_0 belong to the same torsion class then*

$$(1.7.4) \quad \tau_{L_+}^{\varphi'}(t) + (1-t)\tau_{L_0}^{\varphi'}(t) + t\tau_{L_-}^{\varphi'}(t) = 0.$$

1.7.3. Sign-refined torsion and a normalized one variable twisted Alexander function.

1.7.3.1. *A skein relation for sign-refined torsion.* We consider sign-refined torsion, see Section 1.2.6. In all that follow the bases c and \tilde{c} for the chain complexes are

induced from the triangulations of the spaces as previously mentioned at the beginning of Section 1.7.2. There are two cases:

Case 1: The two strands of L_+ at the crossing come from the same component. See Figure 1.11. Suppose that the crossing involves the v th component of L_+ . The bases

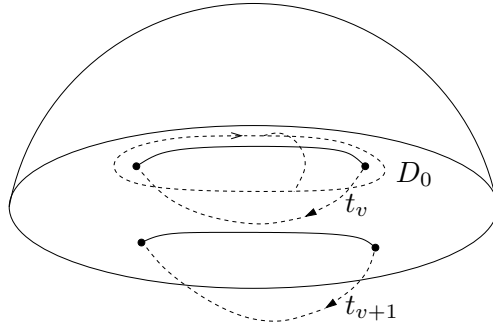


FIGURE 1.11. Case 1.

h_α for $H_*(X_\alpha; \mathbb{R})$, $\alpha = +, -$ consist of $[pt]$, $t_1, \dots, t_v, q_1, \dots, q_{v-1}$, where q_i represents the i th boundary component of L_+ and t_i represent the (oriented) meridian of this component. The basis for $H_*(X_0; \mathbb{R})$ consists of $[pt]$, $t_1, \dots, t_{v+1}, q_1, \dots, q_v$. The basis h_0 for $H_*(V; \mathbb{R})$ consists of $[pt]$, $t_1, \dots, t_{v+1}, q_1, \dots, q_{v-1}$.

We want to compare the terms $\check{\tau}(C_*(X_\alpha, \mathbb{R}), c_\alpha, h_\alpha)$. Consider the short exact sequence of chain complexes:

$$0 \rightarrow C_*(V; \mathbb{R}) \rightarrow C_*(X_\alpha; \mathbb{R}) \rightarrow C_*(X_\alpha, V; \mathbb{R}) \rightarrow 0.$$

Applying the product formula for sign-refined torsion (1.2.7) we obtain

$$\check{\tau}(C_*(X_\alpha; \mathbb{R})) = (-1)^{\mu_\alpha + \nu} \check{\tau}(C_*(V; \mathbb{R})) \check{\tau}(C_*(X_\alpha, V; \mathbb{R})) \tau(\mathcal{H}_\alpha),$$

where \mathcal{H}_α is the long exact homological sequence of the pair (X_α, V) with real coefficients, and

$$\begin{aligned} \mu_\alpha = \sum [& (\beta_i(C_*(X_\alpha; \mathbb{R})) + 1)(\beta_i(C_*(V; \mathbb{R})) + \beta_i(C_*(X_\alpha, V; \mathbb{R}))) + \\ & + \beta_{i-1}(C_*(V; \mathbb{R}))\beta_i(C_*(X_\alpha, V; \mathbb{R}))] \pmod{2} \end{aligned}$$

and

$$\nu = \sum_{i=0}^m \gamma_i(C_*(X_\alpha, V; \mathbb{R}))\gamma_{i-1}(C_*(V; \mathbb{R})).$$

Notice that ν does not depend on α .

Since the term $\check{\tau}(C_*(V; \mathbb{R}))\check{\tau}(C_*(X_\alpha, V; \mathbb{R}))$ does not depend on α we only need to compare $(-1)^{\mu_\alpha}\text{sign}(\tau(\mathcal{H}_\alpha))$. Straightforward calculations show that $\mu_+ \equiv \mu_- \equiv \mu_o + \nu \pmod{2}$. Because $H_2(X_\alpha, V; \mathbb{R}) = 0$, the chain complex \mathcal{H}_α has two portions:

$$0 \rightarrow H_0(V; \mathbb{R}) \rightarrow H_0(X_\alpha; \mathbb{R}) \rightarrow H_0(X_\alpha, V; \mathbb{R}) \rightarrow 0$$

and,

$$0 \rightarrow H_2(V; \mathbb{R}) \rightarrow H_2(X_\alpha; \mathbb{R}) \rightarrow H_2(X_\alpha, V; \mathbb{R}) \rightarrow H_1(V; \mathbb{R}) \rightarrow H_1(X_\alpha; \mathbb{R}) \rightarrow 0.$$

It is clear that for the purpose of comparison we only need to look at the second portion.

When $\alpha = +$: Noting the dimensions of the vector spaces we see that the torsion of \mathcal{H}_+ is the torsion of the chain $0 \rightarrow H_2(X_\alpha, V; \mathbb{R}) \xrightarrow{\partial} H_1(V; \mathbb{R}) \rightarrow H_1(X_\alpha; \mathbb{R}) \rightarrow 0$. Since $[\partial D_+] = [bd] = t_v - t_{v+1}$, it follows that $\tau(\mathcal{H}_+)$ is the determinant of the change of bases matrix $[(t_v - t_{v+1}, t_1, \dots, t_v)/(t_1, \dots, t_v, t_{v+1})]$, which is $(-1)^{v+1}$.

When $\alpha = -$: In this case $[\partial D_-] = [ac] = t_{v+1} - t_v$, thus $\tau(\mathcal{H}_-) = (-1)^v$.

When $\alpha = 0$: The torsion $\tau(\mathcal{H}_0)$ is the torsion of the chain $0 \rightarrow H_2(V; \mathbb{R}) \xrightarrow{i_*} H_2(X_0; \mathbb{R}) \xrightarrow{j_*} H_2(X_0, V; \mathbb{R}) \rightarrow 0$. The map i_* is an injection, $i_*(q_i) = q_i$, $1 \leq i \leq v-1$.

The disk D_0 is a representative of a generator of $H_2(X_0, V; \mathbb{R})$. We need to take a lift of $[D_0]$ under the map j_* . It can be seen that the union of D_0 with part of the boundary of V constitutes either one of the two boundary components of X_0 corresponding to q_v and q_{v+1} (see Figure 1.12). Because of the chosen orientation of

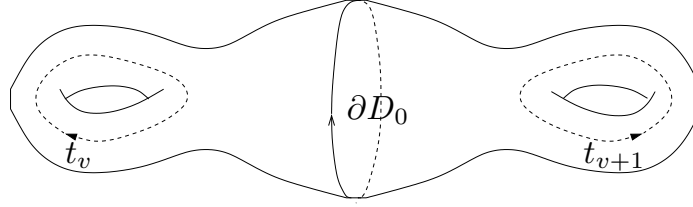


FIGURE 1.12. The disk D_0 .

∂D_0 the two corresponding elements in $H_2(X_0)$, which are lifts of $[D_0]$ under j_* , are $-q_v$ and $q_{v+1} = -(q_1 + q_2 + \dots + q_v)$. The choice of either lift would result that $\tau(\mathcal{H}_0) = [(q_1, \dots, q_{v-1}, -q_v)/(q_1, \dots, q_v)] = -1$.

Collecting the above computations and comparisons of μ_α and $\tau(\mathcal{H}_\alpha)$ we conclude that $\check{\tau}(C_*(X_+, \mathbb{R})) = -\check{\tau}(C_*(X_-, \mathbb{R})) = \check{\tau}(C_*(X_0, \mathbb{R}))$.

Case 2: The two strands of L_+ at the crossing come from different components. See Figure 1.13. Similar to Case 1, the comparison of $\check{\tau}(C_*(X_\alpha; \mathbb{R}))$ is reduced to the

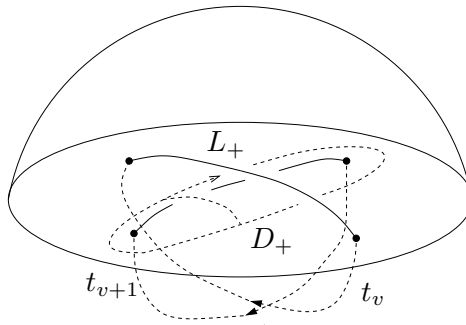


FIGURE 1.13. Case 2.

comparison of $(-1)^{\mu_\alpha} \text{sign}(\tau(\mathcal{H}_\alpha))$. Straightforward calculations give that $\mu_+ \equiv \mu_- \equiv \mu_o + v \pmod{2}$. Again to study $\tau(\mathcal{H}_\alpha)$ we only need to pay attention to the exact

chain

$$0 \rightarrow H_2(V; \mathbb{R}) \rightarrow H_2(X_\alpha; \mathbb{R}) \rightarrow H_2(X_\alpha, V; \mathbb{R}) \rightarrow H_1(V; \mathbb{R}) \rightarrow H_1(X_\alpha; \mathbb{R}) \rightarrow 0.$$

When $\alpha = +$: $\tau(\mathcal{H}_+)$ is the torsion of the chain $0 \rightarrow H_2(V; \mathbb{R}) \rightarrow H_2(X_+; \mathbb{R}) \rightarrow H_2(X_+, V; \mathbb{R}) \rightarrow 0$. The lift of $[D_+] \in H_2(X_+, V; \mathbb{R})$ to $H_2(X_+; \mathbb{R})$ is either q_v or $-q_{v+1}$. With either lift the we have $\tau(\mathcal{H}_+) = [(q_1, \dots, q_{v-1}, q_v)/(q_1, \dots, q_v)] = 1$.

When $\alpha = -$: Just as the case $\alpha = +$, except that now the lift of $[D_-]$ can be either $-q_v$ or q_{v+1} , so $\tau(\mathcal{H}_-) = -1$.

When $\alpha = 0$: $\tau(\mathcal{H}_0)$ is the torsion of the chain $0 \rightarrow H_2(X_0, V; \mathbb{R}) \xrightarrow{\partial} H_1(V; \mathbb{R}) \rightarrow H_1(X_0; \mathbb{R}) \rightarrow 0$. Since $[\partial D_0] = [a^{-1}b] = t_{v+1} - t_v \in H_1(V; \mathbb{R})$ we have $\tau(\mathcal{H}_0) = [(t_{v+1} - t_v, t_1, \dots, t_v)/(t_1, \dots, t_{v+1})] = (-1)^v$.

We conclude that, as in Case 1, $\check{\tau}(C_*(X_+, \mathbb{R})) = -\check{\tau}(C_*(X_-, \mathbb{R})) = \check{\tau}(C_*(X_0, \mathbb{R}))$.

Now Formula (1.2.4) and the skein relation for unrefined torsion (1.7.4) give us a skein relation for sign-refined torsion:

$$(1.7.5) \quad \tau_{0, L_+}^{\varphi'}(t) + (1-t)\tau_{0, L_0}^{\varphi'}(t) - t\tau_{0, L_-}^{\varphi'}(t) = 0,$$

provided that L_+ , L_- and L_0 belong to the same torsion class.

1.7.3.2. *Definition of the normalized one variable twisted Alexander function.* Now using sign-refined torsion we will define a normalized one variable twisted Alexander function.

For a given link L the sign-refined torsion $\tau_{0, L}^{\varphi'}(t)$ is defined up to t^n , $n \in \mathbb{Z}$. Using Theorem 1.7.3, there is a number $r \in \mathbb{Z}$ arising from the symmetry of (un-refined) Reidemeister torsion such that $\tau_{0, L}^{\varphi'}(t^{-1}) = \pm t^r \tau_{0, L}^{\varphi'}(t)$ as elements in $\mathbb{Q}(t)$.

Define the normalized twisted Alexander function of a link L to be

$$(1.7.6) \quad \nabla_L(t) = -t^r \tau_{0, L}^{\varphi'}(t^2).$$

Notice that $\nabla_L(t^{-1}) = -t^{-r}\tau_{0,L}^{\varphi'}(t^{-2}) = \pm t^{-r}t^{2r}\tau_{0,L}^{\varphi'}(t^2) = \pm t^r\tau_{0,L}^{\varphi'}(t^2) = \pm\nabla_L(t)$. Thus $\nabla_L(t)$ is symmetric, up to a sign. From Theorems 1.7.2 and 1.7.1, the function $\nabla_L(t)$ is an element of $\mathbb{Z}[t^{\pm 1}]$ (a Laurent polynomial) if L is nontorsion, and is an element of $\mathbb{Z}[t^{\pm 1}, (t - t^{-1})^{-1}]$ (a Laurent polynomial divided by $(t - t^{-1})^n$) if L is torsion.

ASSERTION 1.7.6. *The function $\nabla_L(t)$ does not depend on the choice of a representative of $\tau_{0,L}^{\varphi'}(t)$.*

PROOF. Suppose that τ and τ' are two representatives of the (sign-refined) torsion $\tau_{0,L}^{\varphi'}$. Then $\tau'(t) = t^m\tau(t)$ for some $m \in \mathbb{Z}$. This implies that there is some $n \in \mathbb{Z}$ such that $\nabla'(t) = t^n\nabla(t)$. Since $\nabla(t^{-1}) = \pm\nabla(t)$ and $\nabla'(t^{-1}) = \pm\nabla'(t)$ we must have $n = 0$, that is $\nabla'(t) = \nabla(t)$. \square

REMARK 1.7.7. Our $\tau_{0,L}^{\varphi'}(t)$ plays a rather similar role to Turaev's Alexander function as in [Tur02b, p. 86].

1.7.4. A skein relation for the normalized twisted Alexander function.

THEOREM 1.7.8. *If L_+ , L_- and L_0 belong to the same torsion class then the normalized one variable twisted Alexander function satisfies the skein relation:*

$$\nabla_{L_+}(t) - \nabla_{L_-}(t) = (t - t^{-1})\nabla_{L_0}(t).$$

PROOF. Replacing t by t^2 in Equation (1.7.5), and using Equation (1.7.6) we have:

$$t^{-r_+}\nabla_{L_+}(t) + (1 - t^2)t^{-r_0}\nabla_{L_0}(t) - t^{2-r_-}\nabla_{L_-}(t) = 0,$$

that is

$$\nabla_{L_+}(t) = (t - t^{-1})t^{1+r_+ - r_0}\nabla_{L_0}(t) + t^{2+r_+ - r_-}\nabla_{L_-}(t).$$

Let $u = 2 + r_+ - r_-$ and $v = 1 + r_+ - r_0$ we get

$$(1.7.7) \quad \nabla_{L_+}(t) = (t - t^{-1})t^v \nabla_{L_0}(t) + t^u \nabla_{L_-}(t).$$

The purpose of the rest of the proof is to show that $u = v = 0$. The idea is to show that u and v are independent of the link. This is achieved by studying the numbers r_α . Since these numbers arise from the symmetry of torsion, a study of duality of torsion is needed.

Topologically, the complement X_α of L_α is the union of V and a 2-handle H_α glued to V along the loop ∂D_α . Assume that X_α is triangulated by a triangulation of V together with a compatible triangulation of H_α . Let \tilde{X}_α be the $D = \mathbb{Z} \times \text{Tors } H_1(X_\alpha)$ cover of X_α corresponding to the kernel of the map $proj_\alpha : \pi_1(X_\alpha) \rightarrow H_1(X_\alpha) \rightarrow G \times \text{Tors } H_1(X_\alpha) \rightarrow \{t^m : m \in \mathbb{Z}\} \times \text{Tors } H_1(X_\alpha)$. As in Section 1.7.2.2, \tilde{X}_α can be constructed as $\tilde{V} \cup_{t \in D} t\tilde{H}_\alpha$, that is \tilde{V} with disjoint copies of H_α glued in along the lifts of ∂D_α . Because of our assumption that L_α belong to the same torsion class, the deck transformation group D does not depend on α . An induced triangulation Y_α of \tilde{X}_α is obtained, which is equivariant under the action of D . Let Y_α^* be its dual cell decomposition and ∂Y_α^* be the restriction of Y_α^* to the boundary $\partial \tilde{X}_\alpha$.

$$\text{Let } E_\alpha = \mathbb{Q}(t) \otimes_{\varphi'} C_*(Y_\alpha), F_\alpha = \mathbb{Q}(t) \otimes_{\varphi'} C_*(Y_\alpha^*), \partial F_\alpha = \mathbb{Q}(t) \otimes_{\varphi'} C_*(\partial Y_\alpha^*).$$

Choose a fundamental family of cells e_α for Y_α such that all the cells in e_α that cover a cell in H_α are contained in the same \tilde{H}_α . Denote by e_α^* the family of cells in Y_α^* that are dual to the simplexes in e_α .

The proof consists of the following steps.

Step 1: Studying $\tau(F_\alpha)$. The triangulation Y_α and its dual cell decomposition Y_α^* has a common cellular subdivision, namely the first barycentric subdivision Y'_α of Y_α . It is possible to choose two fundamental family of cells for \tilde{X}_α corresponding to Y'_α . The first is a , consisting of the cells a_1, a_2, \dots, a_n , each of which is contained in a cell

in e_α . This provides a chosen basis for the chain complex of vector spaces E_α . The second fundamental family of cells is b , consisting of the cells b_1, b_2, \dots, b_n , each of which is contained in a cell in e_α^* , providing a chosen basis for the chain complex F_α .

Using invariance of torsion under cellular subdivision (see [Tur86, Lemma 4.3.3 iii]) we have $\tau(E_\alpha, e_\alpha) = \pm\tau(\mathbb{Q}(t) \otimes_{\varphi'} C_*(Y'_\alpha), a)$ and $\tau(F_\alpha, e_\alpha^*) = \pm\tau(\mathbb{Q}(t) \otimes_{\varphi'} C_*(Y'_\alpha), b)$. Let us compare the torsion of the same chain complex $\mathbb{Q}(t) \otimes_{\varphi'} C_*(Y'_\alpha)$ with different bases a and b .

We have $\tau(\mathbb{Q}(t) \otimes_{\varphi'} C_*(Y'_\alpha), b) = \tau(\mathbb{Q}(t) \otimes_{\varphi'} C_*(Y'_\alpha), a)\varphi'([b/a])$, where $[b/a] \in D$ denotes the determinant of the change of base matrix. If two cells a_i and b_j cover the same cell in the 2-handle H_α then they must be contained in the same \tilde{H}_α because of our choice for e_α above, and so a_i and b_j must be the same cell. This means that the correctional term $\varphi'([b/a])$ does not depend on α .

Thus there is $\beta \in \mathbb{Z}$ which does not depend on α such that

$$(1.7.8) \quad \tau(F_\alpha, e_\alpha^*) = \pm t^\beta \tau(E_\alpha, e_\alpha) = \pm t^\beta \tau_{L_\alpha}^{\varphi'}(t).$$

Step 2: Studying the chain ∂F_α . Consider the short exact sequence of chain complexes

$$(1.7.9) \quad 0 \rightarrow \partial F_\alpha \rightarrow F_\alpha \rightarrow F_\alpha / \partial F_\alpha \rightarrow 0.$$

Note that ∂X_α is a collection of tori.

ASSERTION 1.7.9. *The chain ∂F_α is exact and its torsion – the torsion of a collection of tori – is 1 up to $\pm t^n, n \in \mathbb{Z}$.*

PROOF. This is essentially because the torsion of a torus is 1, see Example 1.2.6. We only need to show that the associated chain complex there is always acyclic. The contrary only happens when both the meridian and the longitude of the torus T are killed in the inclusion $H_1(T) \rightarrow H_1(X_\alpha)$, and thus the same is true for the whole

$H_1(T)$. This is not case because of, for example, the fact that with coefficients in a field, only half of the first homology group of the boundary is killed in the inclusion to the homology group of the 3-manifold. Here is another, direct argument. Let ℓ and m be the longitude and the meridian of the torus boundary. Suppose that one of them is killed in the inclusion $i_* : H_1(T) \rightarrow H_1(X_\alpha)$, for example ℓ . Then $\ell = \partial c$, where c is a singular surface in X_α , representing an element of $H_2(X_\alpha, T)$. It is clear that the intersection number between $[c]$ and $[m]$ is 1, so $[m] \neq 0$. \square

The long homological exact sequence associated with the short exact sequence (1.7.9) above shows that F_α is exact if and only if $F_\alpha/\partial F_\alpha$ is exact. Note that by the invariance of torsion under cellular subdivisions, F_α is exact if and only if E_α is exact, and in any case $\tau(F_\alpha) = \tau(E_\alpha)$ up to $\pm t^n$, $n \in \mathbb{Z}$. The product formula for torsion of chain complexes applied to the short exact sequence (1.7.9) gives

$$(1.7.10) \quad \tau(F_\alpha) = \pm \tau(\partial F_\alpha) \tau(F_\alpha/\partial F_\alpha).$$

Both sides are zero when F_α is not exact.

Recall that ∂X_α is a collection of tori. Let R be the union of those tori which do not involve at the crossing, i.e. $R \cap B = \emptyset$, where B is the ball enclosing the crossing under scrutiny as in Figure 1.8. Then $\partial X_\alpha \setminus R$ is a disjoint union of two tori if the two strands at the crossing belong to different components of the link or it is just a torus if the two strands belong to the same component.

Let $P = \mathbb{Q}(t) \otimes_{\varphi'} C_*(\partial Y^*|_R)$ and $Q_\alpha = \mathbb{Q}(t) \otimes_{\varphi'} C_*(\partial Y^*|_{\partial X_\alpha \setminus R})$. Then $\partial F_\alpha = P \oplus Q_\alpha$. Note that ∂F_α , P and Q_α are all acyclic chain complexes. The torsion of P does not depend on α and is 1 up to units: $\tau(P) = \pm t^p$ for some $p \in \mathbb{Z}$, on the other hand $\tau(Q_\alpha) = \pm t^{q_\alpha}$ for some $q_\alpha \in \mathbb{Z}$. The number q_α depends on how the lifting cells are chosen. It depends only on whether the two strands at the crossing under

investigation belong to the same component or two different components of the link L_α . The product formula for torsion gives us $\tau(\partial F_\alpha) = \pm\tau(P)\tau(Q_\alpha) = \pm t^{p+q_\alpha}$.

Step 3: Studying $\tau(F_\alpha/\partial F_\alpha)$. According to Duality, $\tau(F_\alpha/\partial F_\alpha) = \overline{\tau(E_\alpha)} = \tau_{L_\alpha}^{\varphi'}(t^{-1}) = \pm t^{r_\alpha} \tau_{L_\alpha}^{\varphi'}(t)$. Note that this r_α is the one in Equation (1.7.6).

Step 4: Skein relation for ∇ . From Equation (1.7.10), Step 2 and Step 3 we have $\tau(F_\alpha) = \pm t^{p+q_\alpha} t^{r_\alpha} \tau_{L_\alpha}^{\varphi'}(t)$. Comparing with Equation (1.7.8) we get $\pm t^{p+q_\alpha+r_\alpha} \tau_{L_\alpha}^{\varphi'}(t) = \pm t^\beta \tau_{L_\alpha}^{\varphi'}(t)$. This gives us

$$(1.7.11) \quad \beta = \beta_\alpha = p + q_\alpha + r_\alpha.$$

Using Equation (1.7.11) we have $u = 2 + r_+ - r_- = 2 + q_- - q_+$ and $v = 1 + r_+ - r_0 = 1 + q_0 - q_+$. Thus Equation (1.7.7) depends on the links L_α only to the extent that whether the two strands at the crossing under investigation belong to the same component or two different components of the link L_α . Equation (1.7.7) is satisfied with the same u and v for all link L_+ whose two strands at the crossing come from the same component, and is also satisfied with the same u and v for all link L_+ whose two strands at the crossing come from two different components. Thus in each case a particular example is enough to determine the values of u and v .

Case 1: The two strands of L_+ at the crossing come from one component. Consider the knot 3_1 and the particular crossing in Figure 1.14. Direct computation shows

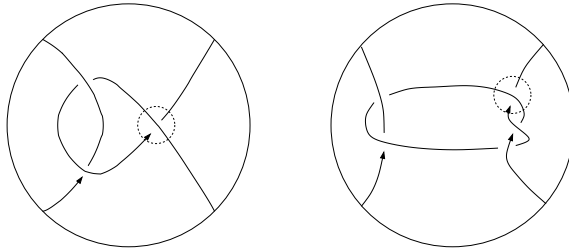


FIGURE 1.14. The knots 3_1 and 5_6 .

that $\nabla_{L_+}(t) = \pm(t - t^{-1})$, $\nabla_{L_-}(t) = \pm(t - t^{-1})$, and $\nabla_{L_0}(t) = 0$, thus $u = 0$.

Also consider the knot 5_6 in that figure. We have $\nabla_{L_+}(t) = \pm(t - t^{-1})$, $\nabla_{L_-}(t) = \pm(t - t^{-1})(t^2 - 1 + t^{-2})$, and $\nabla_{L_0}(t) = \pm(t - t^{-1})^2$, thus $v = 0$.

Case 2: The two strands of L_+ at the crossing come from two different components.

Consider the link 4_2^2 in Figure 1.15. At the first crossing in the figure, $\nabla_{L_+}(t) =$

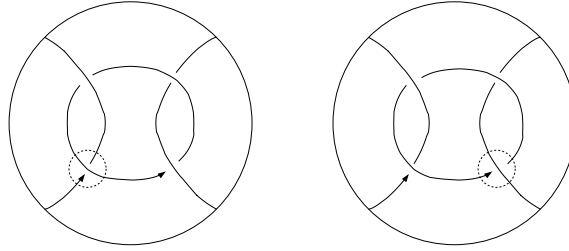


FIGURE 1.15. The link 4_2^2 .

$\pm(t - t^{-1})^2$, $\nabla_{L_0}(t) = \pm(t - t^{-1})$, and $\nabla_{L_-}(t) = 0$, thus $v = 0$. On the other hand at the second crossing in the figure $\nabla_{L_-}(t) = \pm(t - t^{-1})^2$, $\nabla_{L_0}(t) = \pm(t - t^{-1})$, and $\nabla_{L_+}(t) = 0$, thus $u = 0$.

In both cases $u = v = 0$, and the proof of Theorem 1.7.8 is completed. □

1.8. Relationships among twisted and untwisted Alexander polynomials

Suppose that L is a nontorsion link in $\mathbb{R}P^3$. Let \tilde{L} be the preimage of L under the canonical covering map from S^3 to $\mathbb{R}P^3$. Because each component of L is null-homologous hence is null-homotopic in $\mathbb{R}P^3$, its preimage in S^3 has two components. Thus \tilde{L} has an even number of components. A way to draw a diagram for the lift \tilde{L} is to put two parallel copies of L , one on the top disk and another one on bottom disk of a cylinder, then connect the corresponding points on the boundary circles of the disks by vertical lines. For example, the lift of the knot 2_1 in Drobtukhina's table (see Figure 1.7) is the link 4_1^2 in Rolfsen's table.

Let X be the complement of L in $\mathbb{R}P^3$, let $H = H_1(X) = G \times \mathbb{Z}_2$, where G is the free part, and the torsion part is generated by u , as in Section 1.5.1.

Consider the following diagram of coverings:

$$(1.8.1) \quad \begin{array}{ccccc} & & \tilde{X} & & \\ & \swarrow^{p_2} & \downarrow p & \searrow^{p_4} & \\ & \mathbb{Z}_2 & G \times \mathbb{Z}_2 & G & \\ \tilde{X}_G & & & & \tilde{X}_2 = S^3 \setminus \tilde{L} \\ & \searrow_{p_1} & & \swarrow_{p_3} & \\ & G & & \mathbb{Z}_2 & \\ & & X = \mathbb{RP}^3 \setminus L & & \end{array}$$

In the diagram $p : \tilde{X} \rightarrow X$ corresponds to the kernel of the map $\pi_1(X) \rightarrow H$; $p_1 : \tilde{X}_G \rightarrow X$ corresponds to the kernel of the map $\pi_1(X) \rightarrow H \rightarrow G$; $p_3 : \tilde{X}_2 \rightarrow X$ corresponds to the kernel of the map $\pi_1(X) \rightarrow H \rightarrow \mathbb{Z}_2$; and p_2 and p_4 are lifts of p . The diagram is commutative. The cellular structure of X induces cellular structures on the remaining spaces.

Let $C_i^+(\tilde{X})$ be the subcomplex of $C_i(\tilde{X})$ generated by chains of the form $\sigma + u\sigma$ where σ is an i -cell in \tilde{X} . Similarly let $C_i^-(\tilde{X})$ be the subcomplex generated by chains of the form $\sigma - u\sigma$. Consider $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_i(\tilde{X})$, where φ is the twisted map of Section 1.5.1.

PROPOSITION 1.8.1. *We have the following isomorphisms of $\mathbb{Q}(G)$ -vector spaces:*

- a). $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i(\tilde{X}) = (\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^+(\tilde{X})) \oplus (\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^-(\tilde{X}))$.
- b). $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i(\tilde{X}_G) \cong \mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^+(\tilde{X})$.
- c). $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_i(\tilde{X}) \cong \mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^-(\tilde{X})$.

PROOF. Here we are dealing with standard homology with local coefficients and the following proof is adapted from Hatcher [**Hat01**, p. 330].

a). Noting that $C_i^+(\tilde{X}) \cap C_i^-(\tilde{X}) = \{0\}$ and $\sigma = ((\sigma + u\sigma) + (\sigma - u\sigma))/2$, the result follows immediately.

b). A cell in \tilde{X} is a lift of a cell in \tilde{X}_G . The isomorphism is induced from the map $\sigma \mapsto (\tilde{\sigma} + u\tilde{\sigma})$.

c). Consider the the projection $pr : \mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i(\tilde{X}) \rightarrow \mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_i(\tilde{X})$ mapping $1 \otimes \sigma$ to $1 \otimes_{\varphi} \sigma$. We have $pr(1 \otimes (\sigma + u\sigma)) = 1 \otimes_{\varphi} \sigma + 1 \otimes_{\varphi} u\sigma = 1 \otimes_{\varphi} \sigma + 1 \cdot u \otimes_{\varphi} \sigma = 1 \otimes_{\varphi} \sigma + \varphi(u) \otimes_{\varphi} \sigma = 0$, since $\varphi(u) = -1$. This implies $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^+(\tilde{X}) \subset \ker(pr)$. By a similar argument we see that $(\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^-(\tilde{X})) \cap \ker(pr) = \{0\}$. Thus $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^+(\tilde{X}) = \ker(pr)$ and using a) the result follows. \square

Note that $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i(\tilde{X}_G)$ is in fact $\mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C_i(\tilde{X}_G)$. It follows from this proposition that we have the short exact sequence of chain complexes of $\mathbb{Q}(G)$ -vector spaces:

$$(1.8.2) \quad 0 \rightarrow \mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_*(\tilde{X}) \rightarrow \mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_*(\tilde{X}) \rightarrow \mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C_*(\tilde{X}_G) \rightarrow 0.$$

From this sequence we now derive a relationship among multi-variable Alexander polynomials. If L is a nontorsion link having v components then \tilde{L} has $2v$ components. We enumerate so that the i th component and the $(v+i)$ th component of \tilde{L} are projected to the same i th component of L . Let ψ be the homomorphism from $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_v^{\pm 1}, t_{v+1}^{\pm 1}, \dots, t_{2v}^{\pm 1}]$ to $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_v^{\pm 1}]$ identifying t_{v+i} with t_i for all $1 \leq i \leq v$. Consider the multi-variable Alexander polynomial of \tilde{L} , $\Delta_{\tilde{L}}(t_1, t_2, \dots, t_{2v})$. Let $\Delta'_{\tilde{L}}(t_1, t_2, \dots, t_v)$ be obtained from $\Delta_{\tilde{L}}(t_1, t_2, \dots, t_{2v})$ by identifying the t_i and t_{v+i} variables for all $1 \leq i \leq v$, that is $\Delta'(\tilde{L}) = \psi(\Delta(\tilde{L}))$. Recall from our fixed splitting of $H_1(X)$ in Section 1.5.1 that the free part G is generated by the meridians of the components of L , thus $\mathbb{Z}[G] = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_v^{\pm 1}]$.

EXAMPLE 1.8.2. Let K be the knot 2_1 in Drobotukhina's table (see Example 1.5.3). Then $\Delta_K^{\varphi}(t) = t - 1$ and $\Delta_K(t) = t^2 + 1$. The lift \tilde{K} of this knot is the link 4_1^2 in Rolfsen's table, and $\Delta_{\tilde{K}}(t_1, t_2) = t_1 t_2 + 1$, so $\Delta'_{\tilde{K}}(t) = t^2 + 1$.

We have the following relationship among the Alexander polynomial and the twisted Alexander polynomial of a nontorsion link L , and the Alexander polynomial of its preimage \tilde{L} in S^3 .

THEOREM 1.8.3. *If L has one component (a knot) then $(t-1)\Delta'(\tilde{L}) = \Delta(L)\Delta^\varphi(L)$ as elements in $\mathbb{Z}[t^{\pm 1}]$. If L has at least two components then $\Delta'(\tilde{L}) = \Delta(L)\Delta^\varphi(L)$ as elements in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_v^{\pm 1}]$.*

PROOF. Recall the diagram of covering spaces (1.8.1). The map p_4 corresponds to the kernel of the canonical projection $\pi_1(\tilde{X}_2) \rightarrow H_1(\tilde{X}_2) = \langle t_1, \dots, t_{2v}/t_i t_j = t_j t_i \rangle \rightarrow G = \langle t_1, \dots, t_v/t_i t_j = t_j t_i \rangle$, where the second projection identifies t_i and t_{v+i} for all $1 \leq i \leq v$. Thus $p_{4*}(\pi_1(\tilde{X}))$ will be the subgroup of $\pi_1(\tilde{X}_2)$ whose projection to $H_1(\tilde{X}_2)$ is $\{t_1^{\alpha_1} \cdots t_{2v}^{\alpha_{2v}} / \alpha_i + \alpha_{v+i} = 0, 1 \leq i \leq v\}$. Then p_{3*} will send $p_{4*}(\pi_1(\tilde{X}))$ to the subgroup of $\pi_1(X)$ whose projection to $H = H_1(X)$ is $\{t_1^{\alpha_1 + \alpha_{v+1}} \cdots t_v^{\alpha_v + \alpha_{2v}}\} = \{1\}$. So $(p_3 \circ p_4)_*$ sends $\pi_1(\tilde{X})$ to the subgroup of $\pi_1(X)$ which vanishes in $H_1(X)$, this is why $p_3 \circ p_4 = p$.

Now we look at the space \tilde{X} as the G -cover of \tilde{X}_2 corresponding to p_4 . Then there is an action of G on $C_i(\tilde{X})$ turning it to a $\mathbb{Z}[G]$ -module $C'_i(\tilde{X})$, and so we can form the vector space $\mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C'_i(\tilde{X})$. We can see that $\mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C'_i(\tilde{X}) \cong \mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_*(\tilde{X})$. Thus the sequence (1.8.2) becomes

$$0 \rightarrow \mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_*(\tilde{X}) \rightarrow \mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C'_*(\tilde{X}) \rightarrow \mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C_*(\tilde{X}_G) \rightarrow 0.$$

Apply the product formula for torsion (1.2.5) to this short exact sequence we obtain

$$(1.8.3) \quad \tau(\mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C'_*(\tilde{X})) = \pm \tau(\mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C_*(\tilde{X}_G)) \tau(\mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_*(\tilde{X})).$$

Theorem 1.6.1 says that $\tau(\mathbb{Q}(G) \otimes_{\mathbb{Z}[H], \varphi} C_*(\tilde{X}))$ is $\Delta^\varphi(L)$; Remark 1.6.4 says that $\tau(\mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C_*(\tilde{X}_G))$ is $\Delta(L)$ if L has more than one component and is $\Delta(L)/(t-1)$ if L has one component. Finally the identification of torsion and Alexander polynomial

for links in S^3 ([Mil62], [Tur01, p. 55]) says that $\tau(\mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C'_*(\tilde{X}))$ is $\psi(\Delta(\tilde{L}))$ if \tilde{L} has more than one component (here the functority of torsion [Tur01, Lemma 13.5] is used). The theorem then follows from (1.8.3). \square

REMARK 1.8.4. A similar result also holds true if we consider only one variable polynomials.

Finally we include here a proposition related to this topic.

PROPOSITION 1.8.5. *A torus link in S^3 covers a nontorsion link in \mathbb{RP}^3 (under the usual projection) if and only if it has an even number of components.*

PROOF. Suppose that L is a nontorsion link in \mathbb{RP}^3 , covered by \tilde{L} , a $T(m, n)$ torus link in S^3 . Each component of L is null-homotopic in \mathbb{RP}^3 , so its lift is a two component link in S^3 . Thus the number of components $c(\tilde{L})$ of \tilde{L} is even.

Conversely, suppose that \tilde{L} is a $T(m, n)$ torus link in S^3 and $c(\tilde{L})$ is even. Since $c(\tilde{L}) = \gcd(m, n)$, both m and n are even. According to [Chb97, Chb03], a $T(m, n)$ torus link covers a link in \mathbb{RP}^3 if (and only if) 2 divides $m - n$, thus \tilde{L} covers a link L . Since each component of \tilde{L} is null-homotopic in S^3 , its projection is also null-homotopic, hence L is nontorsion. Note also that $c(L) = \frac{1}{2}c(\tilde{L})$. \square

CHAPTER 2

Twisted Alexander polynomial and the A -polynomial of 2-bridge knots

2.1. Background and conventions

Throughout this and the next chapter we consider knots in S^3 .

2.1.1. Representation variety. Let K be a knot in S^3 and $X = S^3 \setminus K$ be its complement. Let $\pi = \pi_1(X)$ be the fundamental group of the complement. Let $R(\pi) = \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ be the set of representations of π to $\text{SL}(2, \mathbb{C})$. This is a complex affine algebraic set, which is called the *representation variety*, although it might be a union of a finite number of (irreducible) algebraic varieties in the sense of algebraic geometry. The group $\text{SL}(2, \mathbb{C})$ acts on $R(\pi)$ by conjugation. The algebro-geometric quotient of $R(\pi)$ under this action is called the *character variety* of π , denoted by $X(\pi)$. The character of a representation ρ is the map $\chi_\rho : \pi \rightarrow \mathbb{C}$ determined by $\chi_\rho(\gamma) = \text{tr } \rho(\gamma)$, for $\gamma \in \pi$. There is a bijection between $X(\pi)$ and the set of characters of representations of π .

2.1.2. The A -polynomial. Let $B = (\mu, \lambda)$ be a pair of meridian-longitude of the boundary torus of X . Let R_U be the subset of $R(\pi)$ containing all representations ρ such that $\rho(\mu)$ and $\rho(\lambda)$ are upper triangular matrices:

$$(2.1.1) \quad \rho(\mu) = \begin{pmatrix} M & * \\ 0 & M^{-1} \end{pmatrix}, \quad \rho(\lambda) = \begin{pmatrix} L & * \\ 0 & L^{-1} \end{pmatrix}$$

(any representation can be conjugated to have this form). Then R_U is an algebraic set, because we only add the requirement that the lower left entries of $\rho(\mu)$ and $\rho(\lambda)$ are

zeros. Define the projection map $\xi : R_U \rightarrow \mathbb{C}^2$ by $\xi(\rho) = (L, M)$. Consider the Zariski closure $\overline{\xi(R_U)}$ of the projection $\xi(R_U) \subset \mathbb{C}^2$. It is known that $\overline{\xi(R_U)}$ is an algebraic set whose components have dimensions zero or one. If a component has dimension one then it is a curve defined by a single polynomial in L and M . The product of these polynomials, divided by $L - 1$, is called the *A-polynomial* of K ¹. The reason for dividing by $L - 1$ is as follows. If ρ is an abelian representation then it factors through $H_1(X) = \langle \mu \rangle$, so $\rho(\lambda)$ is the identity matrix, therefore the component of $\overline{\xi(R_U)}$ corresponding to abelian representations is defined by a single equation $L = 1$. Thus in the construction of the *A-polynomial* one can restrict to nonabelian representations.

It is known that a multiple constant can be chosen so that the *A-polynomial* is an integer polynomial. We assume that the *A-polynomial* has no repeated factors; and that it has no integer factors, i.e. its coefficients are coprime. If instead of the basis $B = (\mu, \lambda)$ we choose the other basis (μ^{-1}, λ^{-1}) then the pair (L, M) is replaced by the pair (L^{-1}, M^{-1}) as can be seen from (2.1.1), and it is known that $A_K(L^{-1}, M^{-1}) = \pm L^m M^n A_K(L, M)$. Thus $A_K(L, M)$ is an integer polynomial defined up to a factor $\pm L^m M^n$.

With finitely many exceptions, corresponding to a pair (L, M) satisfying $A(L, M) = 0$ there is a nonabelian representation $\rho \in R(\pi)$ for which (2.1.1) holds.

For more on the *A-polynomial* we refer to [CCG⁺94], [CL96] and [CL98].

2.1.3. 2-bridge knots. Let $p = 2n + 1$, $n \geq 1$, and $0 < q < p$. The fundamental group of the complement X of the 2-bridge knot $\mathfrak{b}(p, q)$ has a presentation $\pi = \pi_1(X) = \langle a, b/wa = bw \rangle$, where both a and b are meridians. The word w has the form $a^{\epsilon_1} b^{\epsilon_n} a^{\epsilon_2} b^{\epsilon_{n-1}} \cdots a^{\epsilon_n} b^{\epsilon_1}$, where $\epsilon_i = (-1)^{\lfloor iq/p \rfloor} = \pm 1$. In particular, w is palindromic. For example, $\mathfrak{b}(2n + 1, 1)$ is the torus knot $T(2, 2n + 1)$, and in this case $w = (ab)^n$.

¹The letter *A* stands for *affine* – according to Garoufalidis.

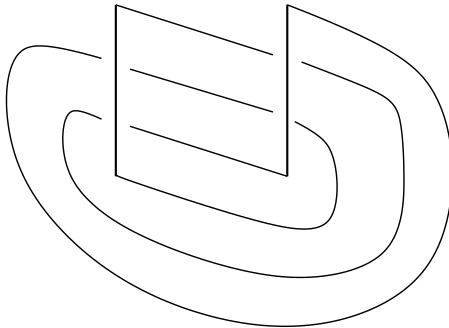


FIGURE 2.1. The trefoil as the 2-bridge knot $\mathfrak{b}(3, 1)$.

We adopt the convention that if $\rho \in R(\pi)$ and x is a word then we write $\text{tr } x$ for $\text{tr } \rho(x)$. Let $x = \text{tr } a$ and $y = \text{tr } ab$. Thang Le [Le93] showed that the character variety $X^{nab}(\pi)$ of nonabelian representations of π is determined by the polynomial $\Phi_{(p,q)}(x, y) = \text{tr } w - \text{tr } w' + \cdots + (-1)^{n-1} \text{tr } w^{(n-1)} + (-1)^n$, here if x is a word then x' denotes the word obtained from x by deleting the two letters at the two ends.

For more on 2-bridge knots see [BZ03], and for representations of 2-bridge knot groups we refer to [Ril84] and [Le93].

2.1.4. Nonabelian and irreducible representations. A representation ρ is said to be *reducible* if the action (i.e. the linear map) it induces on \mathbb{C}^2 fix a one dimensional subspace of \mathbb{C}^2 . This is equivalent to saying that ρ can be conjugated to be a representation by upper triangular matrices (one can take an eigenvector of the linear map as a new basis vector for \mathbb{C}^2). Otherwise ρ is said to be *irreducible*.

An elementary argument (as suggested above) would show that if ρ is irreducible then it is nonabelian. For 2-bridge knots we have a stronger result ([Le93]): Except finitely many cases, a nonabelian representation is irreducible. The Zariski closure $\overline{X^{irr}(\pi)}$ of the set of characters of irreducible representations is exactly the character variety $X^{nab}(\pi)$ of nonabelian representations. Therefore in some arguments we can consider irreducible representations instead of nonabelian representations.

2.1.5. The A -polynomial of 2-bridge knots. Suppose that ρ is an irreducible representation. After conjugations if necessary we may assume that

$$(2.1.2) \quad \rho(a) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} M & 0 \\ -z & M^{-1} \end{pmatrix}.$$

We have $x = \text{tr } a = M + M^{-1}$ and $z = x^2 - 2 - y$ where $y = \text{tr } ab$. Let $\lambda = w \overleftarrow{w} b^{-2e}$, where \overleftarrow{w} is the word obtained from w by writing the letters in w in reversed order (i.e. by interchanging a and b), and e is the sum of the exponents of the letters in w . Then λ represents the longitude of the boundary torus of the knot complement, and we define $\mathcal{L}(M, y)$ to be the upper left entry of the matrix $\rho(\lambda)$. Then up to a factor of the form an integral power of M , $\mathcal{L}(M, y)$ is a polynomial. Because $x = M + M^{-1}$ we can consider Φ as a function in M and y , up to a factor of the form M to an integral power it is a polynomial. The A -polynomial $A(L, M)$ can be computed by deleting repeated factors from the resultant $\text{Res}(\Phi(M, y), \mathcal{L}(M, y) - L)$, where the resultant is computed with respect to y .

The description above can be implemented for computer calculations.

EXAMPLE 2.1.1. The A -polynomial of $\mathfrak{b}(3, 1)$ (the trefoil) is $LM^6 + 1$, and that of $\mathfrak{b}(5, 3)$ (the figure-8 knot) is $-LM^8 + LM^6 + L^2M^4 + 2LM^4 + M^4 + LM^2 - L$.

For further details on the A -polynomial of 2-bridge knots we refer to [CCG⁺94] and [HS04].

2.1.6. The adjoint representation. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of $\text{SL}(2, \mathbb{C})$ consists of 2×2 matrices with zero traces. Consider the adjoint representation of $\text{SL}(2, \mathbb{C})$, $Ad : \text{SL}(2, \mathbb{C}) \rightarrow \text{Aut}(\mathfrak{sl}_2(\mathbb{C}))$. For $A \in \text{SL}(2, \mathbb{C})$ and $x \in \mathfrak{sl}_2(\mathbb{C})$ we have $Ad_A(x) = Ax A^{-1}$. Since $\mathfrak{sl}_2(\mathbb{C})$ can be identified with \mathbb{C}^3 , Ad_A is a linear map on \mathbb{C}^3 and it turns out that it belongs to $\text{SO}(3, \mathbb{C})$. If $\rho \in R(\pi)$ then the composition $Ad \circ \rho$ is a representation of π to $\text{SO}(3, \mathbb{C})$.

2.2. From the A-polynomial to the twisted Alexander polynomial

DEFINITION 2.2.1. Let $\pi = \langle a, b/r = waw^{-1}b^{-1} = 1 \rangle$. Let ρ be the representation of the free group $\langle a, b \rangle$ defined by the formula

$$(2.2.1) \quad \rho(a) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} M & 0 \\ -z & M^{-1} \end{pmatrix}.$$

Extend the map $Ad \circ \rho$ linearly, and consider M and z as formal variables. The twisted Alexander polynomial $\Delta_K^{Ad}(M, z)$ associated to π is defined by

$$\Delta_K^{Ad}(M, z) = \gcd\{\det(Ad \circ \rho(\partial r/\partial a)), \det(Ad \circ \rho(\partial r/\partial b))\} \in \mathbb{C}[M^{\pm 1}, z^{\pm 1}].$$

It is a polynomial in M and z up to a factor $\pm M^m z^n$.

For each pair (L_0, M_0) such that $A_K(L_0, M_0) = 0$ there is a finite number of numbers $z_i \in \mathbb{C}$ such that both polynomial equations $\Phi(M_0, z_i) = 0$ and $\mathcal{L}(M_0, z_i) = L_0$ are satisfied.

PROPOSITION 2.2.2. *Except for finitely many pairs (L_0, M_0) , if $A_K(L_0, M_0) = 0$ then $\Delta_K^{Ad}(M_0, z_i) = 0$.*

PROOF. Except a finite number of pairs (L_0, M_0) , if $A_K(L_0, M_0) = 0$ then there is an irreducible representation $\rho \in R(\pi)$ for which $\rho(\mu) = \begin{pmatrix} M_0 & * \\ 0 & M_0^{-1} \end{pmatrix}$, $\rho(\lambda) = \begin{pmatrix} L_0 & * \\ 0 & L_0^{-1} \end{pmatrix}$, and

$$(2.2.2) \quad \rho(a) = \begin{pmatrix} M_0 & 1 \\ 0 & M_0^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} M_0 & 0 \\ -z_i & M_0^{-1} \end{pmatrix}.$$

Following a standard argument, the knot complement X is simple homotopic to a 2-dimensional cell complex with one 0-cell, two 1-cells and one 2-cell. Letting \tilde{X} be

the universal cover, we can consider the cochain complex of complex vector spaces:

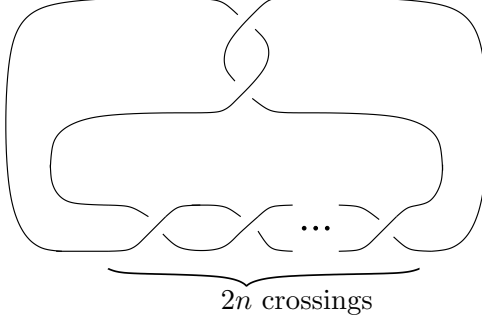
$$0 \leftarrow \mathbb{C}^3 \otimes_{\mathbb{Z}[\pi], Ad \circ \rho} C^2(\tilde{X}) \xleftarrow{\partial_2} \mathbb{C}^3 \otimes_{\mathbb{Z}[\pi], Ad \circ \rho} C^1(\tilde{X}) \xleftarrow{\partial_1} \mathbb{C}^3 \otimes_{\mathbb{Z}[\pi], Ad \circ \rho} C^0(\tilde{X}) \leftarrow 0.$$

Here ∂_2 is represented by the 3×6 -matrix $\begin{pmatrix} Ad \circ \rho(\partial r / \partial a) & Ad \circ \rho(\partial r / \partial b) \end{pmatrix}$ and ∂_1 is represented by the 6×3 -matrix $\begin{pmatrix} Ad \circ \rho(a-1) \\ Ad \circ \rho(b-1) \end{pmatrix}$. A direct computation shows that $Ad \circ \rho(b-1)$ is nonsingular. Thus $\text{rank}(\text{Im } \partial_1) = 3$. The first cohomology group with local coefficients of X is $H_{Ad \circ \rho}^1(X) = \ker \partial_2 / \text{Im } \partial_1$.

At this point we use a theorem of Weil [Wei64] (see [Por97, p. 69], [BZ00]). The theorem asserts that if ρ is an irreducible representation then the Zariski tangent $T_{\chi_\rho}^{Zar}(X(\pi))$ of the character variety $X(\pi)$ at the point χ_ρ is isomorphic as complex vector space to a subspace of the first cohomology group $H_{Ad \circ \rho}^1(X)$. For the Zariski tangent space at a point P of an algebraic variety Y we always have $\text{rank } T_P^{Zar}(Y) \geq \text{rank}(Y)$. In this case because the point χ_ρ arises from a point on the curve defined by $A(L, M)$, the dimension of the irreducible component of $X(\pi)$ containing χ_ρ is at least one (we can also evoke a theorem of Thurston to this effect, see e.g. [CS83, Proposition 3.2.1]). Thus $\text{rank } T_{\chi_\rho}^{Zar}(X(\pi)) \geq 1$, hence $\text{rank } H_{Ad \circ \rho}^1(X) \geq 1$.

Since $\text{rank}(\ker \partial_2 / \text{Im } \partial_1) \geq 1$ and $\text{rank}(\text{Im } \partial_1) = 3$ it follows that $\text{rank}(\ker \partial_2) \geq 4$, hence $\text{rank}(\text{Im } \partial_2) \leq 2$. This means that both 3×3 -matrices $Ad \circ \rho(\partial r / \partial a)$ and $Ad \circ \rho(\partial r / \partial b)$ have ranks less than three and thus are singular. Hence $\det(Ad \circ \rho(\partial r / \partial a)) = \det(Ad \circ \rho(\partial r / \partial b)) = 0$. This means $\Delta_K^{Ad}(M, z)$ vanishes when it is evaluated at (M_0, z_i) . \square

In the special case of a twist knot K_n , which is the 2-bridge knot $\mathfrak{b}(4n+1, 2n+1)$, it is shown in [HS04, p. 203] (note that $K_n = J(2, -2n)$ in their notation) that the correspondence $z_i \mapsto L_0$ is one-to-one. Specifically z can be expressed in terms of L

FIGURE 2.2. The twist knot $K_n, n > 0$.

and M as

$$(2.2.3) \quad z = \frac{(1-L)(1-M^2)}{L+M^2}.$$

Using this change of variable we can write the twisted Alexander polynomial $\Delta_K^{Ad}(M, z)$ as a polynomial $\Delta_K^{Ad}(L, M)$.

THEOREM 2.2.3. *If K is twist knot then the polynomial $A_K(L, M)$ is a factor of the polynomial $\Delta_K^{Ad}(L, M)$.*

PROOF. For a twist knot Proposition 2.2.2 says that the zero set $Z(A)$ of the A -polynomial $A(L, M)$ minus a set I consists of finitely many points is contained in the zero set $Z(\Delta^{Ad})$ of the twisted Alexander polynomial $\Delta^{Ad}(L, M)$. The Zariski closure of $Z(A) \setminus I$ is exactly $Z(A)$. Thus we have $Z(A) \subset Z(\Delta^{Ad})$ and so $A(L, M)$ is a factor of $\Delta^{Ad}(L, M)$. \square

CHAPTER 3

Irreducibility of the A -polynomial of 2-bridge knots

3.1. Introduction

In his recent study on the AJ conjecture which relates the A -polynomial and the colored Jones polynomial of a knot, Thang Le [Le04] proved that for a 2-bridge knot $\mathfrak{b}(p, q)$ the AJ conjecture holds true if the A -polynomial is irreducible and has L -degree $(p-1)/2$. In this chapter we will provide a proof for the result (Theorem 3.2.5 below) that the above condition is satisfied if both p and $(p-1)/2$ are prime and $q \neq 1$.

In a related result, recently Hoste and Shanahan [HS04] using trace field theory have proved that the A -polynomial of the twist knot K_n , which is the 2-bridge knot $\mathfrak{b}(4n+1, 2n+1)$, is irreducible. From their recursive formula it can be checked easily that the L -degree is exactly $2n$.

3.2. Proofs

Let $\Phi_n(x, y) = \Phi_{(p,1)}(x, y)$, where $p = 2n + 1$. It has been shown in [Le93, Proposition 4.3.1] (also see below) that $\Phi_n(x, y)$ does not depend on x .

PROPOSITION 3.2.1. $\Phi_n(y)$ is irreducible if and only if $2n + 1$ is prime.

PROOF. It is immediate from [Le93, Proposition 4.3.1] that $\Phi_n(2y) = (T_n(y) + T_{n+1}(y))/(y+1)$, where T_n is the n th Chebyshev polynomial (of the first kind). Let $\tilde{\Phi}_n(y) = \Phi_n(2y)$. It is well-known that by letting $\theta = \cos y$, we can write $T_n(y) = \cos(n\theta)$, and so $\tilde{\Phi}_n(\theta) = \cos((2n+1)\frac{\theta}{2})/\cos(\frac{\theta}{2})$. It also follows that $\tilde{\Phi}_n(y)$ is an integer polynomial of degree n with exactly n roots given by $y = \cos(\frac{2k+1}{2n+1}\pi)$, $0 \leq k \leq n-1$. Fix $\theta = \pi/p$. Noting that $\tilde{\Phi}_n$ has no integer factor since $\tilde{\Phi}_n(0) = \pm 1$ we see that $\tilde{\Phi}_n$

is irreducible, and so is Φ_n , if and only if the extension field degree $[\mathbb{Q}(\cos \theta) : \mathbb{Q}]$ is exactly the degree of $\tilde{\Phi}_n$.

Noticing that $\cos \theta = (e^{i\theta} + e^{-i\theta})/2$, we want to study the extension field $\mathbb{Q}(e^{i\theta})$. It is well-known (see, e.g. [Lan93, p. 276]) that the irreducible polynomial of $e^{i\theta}$ is the cyclotomic polynomial

$$C_{2p}(y) = \prod_{1 \leq d \leq 2p, (d, 2p)=1} (x - e^{d\pi i/p}).$$

This is an integer polynomial whose degree is $\varphi(2p) = \varphi(p)$, here φ is the Euler totient function. Thus the degree of the extension field is $[\mathbb{Q}(e^{i\theta}) : \mathbb{Q}] = \varphi(p)$. From the identity $(x - e^{\theta i})(x - e^{-\theta i}) = x^2 - 2(\cos \theta)x + 1$, we see that $[\mathbb{Q}(e^{i\theta}) : \mathbb{Q}(\cos \theta)] = 2$, thus $[\mathbb{Q}(\cos \theta) : \mathbb{Q}] = \varphi(p)/2$. Therefore Φ_n is irreducible if and only if $\varphi(p) = p - 1$, which happens if and only if p is prime. \square

PROPOSITION 3.2.2. *We have $\Phi_{(p,q)}(0, y) = \Phi_{(p,1)}(y)$. Hence if $\Phi_{(p,1)}(y)$ is irreducible then $\Phi_{(p,q)}(x, y)$ is also irreducible.*

PROOF. Recall from section 2.1.5 that we can write $\rho(a) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}$ and $\rho(b) = \begin{pmatrix} M & 0 \\ -z & M^{-1} \end{pmatrix}$, where $M + M^{-1} = x$ and $z = x^2 - 2 - y$. If $x = \text{tr } a = \text{tr } b = M + M^{-1} = 0$ then it is immediate that $\rho(a^{-1}) = -\rho(a)$ and $\rho(b^{-1}) = -\rho(b)$ (this can also be seen from the Cayley-Hamilton Theorem: the characteristic polynomial of $\rho(a)$ is $t^2 - (\text{tr } a)t + 1$). Recall that $\Phi_{(p,q)}(x, y) = \text{tr } w - \text{tr } w' + \dots + (-1)^{n-1} \text{tr } w^{(n-1)} + (-1)^n$. Because the word w is palindromic, so is each word $w^{(i)}$, $0 \leq i \leq n - 1$, and hence in $w^{(i)}$ we have a^{-1} and b^{-1} appear in pairs. That means $\rho(w^{(i)})$ does not change if we replace a^{-1} by a and b^{-1} by b . Thus $\rho(w^{(i)}) = \rho((ab)^{n-i})$. Recalling that for a torus knot $\mathfrak{b}(p, 1)$ we have $w = (ab)^n$, the result follows. \square

Because $x = M + M^{-1}$ we can consider Φ as a function in M and y , and it is a polynomial up to a factor of the form M to an integral power, which is omitted.

PROPOSITION 3.2.3. *If $\Phi(M, y)$ is irreducible then $A(L, M)$ is irreducible.*

PROOF. Recall from Section 2.1.5 that the A -polynomial $A(L, M)$ of a 2-bridge knot can be computed by deleting repeated factors from $\text{Res}(\Phi(M, y), \mathcal{L}(M, y) - L)$, where $\mathcal{L}(M, y)$ is a polynomial and the resultant is computed with respect to y .

We have $A(L, M) = 0$ if and only if there is y such that $\Phi(M, y) = 0$ and $\mathcal{L}(M, y) = L$. Writing $Z(f)$ for the zero set of a polynomial f , we see that for each $(M, L) \in Z(A(L, M))$ there is $(M, y) \in Z(\Phi(M, y))$ such that $(M, \mathcal{L}(M, y)) = (M, L)$.

In what follows we use some simple notions in algebraic geometry, which can be found for example in [Har77]. Consider the map $pr : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by $pr(u, v) = (u, \mathcal{L}(u, v))$. This map is continuous under the Zariski topology. It projects $Z(\Phi(M, y))$ onto $Z(A(L, M))$.

Note that f is an irreducible polynomial if and only if $Z(f)$ is an irreducible algebraic set. Now suppose that the A -polynomial is reducible, hence $Z(A(L, M))$ is a union of two nonempty closed subsets B and C . Then $pr^{-1}(B) \cap Z(\Phi)$ and $pr^{-1}(C) \cap Z(\Phi)$ are two nonempty closed sets whose union is $Z(\Phi)$. This implies that $\Phi(M, y)$ is reducible, a contradiction. \square

PROPOSITION 3.2.4. *If the L -degree of $A(L, M)$ is 1 then $q = 1$, and so $\mathfrak{b}(p, q)$ is the torus knot $T(2, p)$.*

The idea for the following proof was communicated to us by Nathan Dunfield. We also thank Xingru Zhang for a discussion on this topic.

PROOF. We need the concept of Newton polygons of A -polynomials. The Newton polygon of $A(L, M)$ is the convex hull of the set of points (i, j) on the real LM -plane such that the coefficient a_{ij} of the term $a_{ij}L^iM^j$ of $A(L, M)$ is nonzero. The slopes

of the sides of the Newton polygon are boundary slopes of incompressible surfaces in the knot complement ([**CCG⁺94**]).

For example the following figure shows the Newton polygon of the torus knot $\mathfrak{b}(3, 1) = T(2, 3)$ (the trefoil) whose A -polynomial is $LM^6 + 1$, and that of $\mathfrak{b}(5, 3)$ (the figure-8 knot) whose A -polynomial is $-LM^8 + LM^6 + L^2M^4 + 2LM^4 + M^4 + LM^2 - L$.

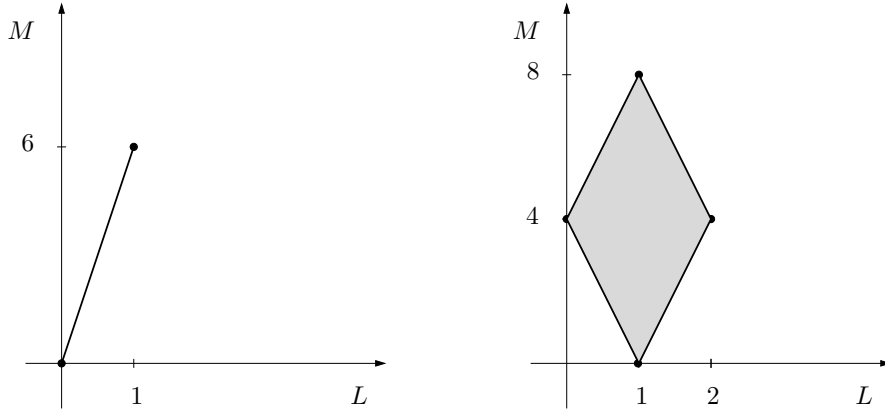


FIGURE 3.1. Newton polygons of the A -polynomials of $\mathfrak{b}(3, 1)$ and $\mathfrak{b}(5, 3)$.

Suppose that the L -degree of $A(L, M)$ is 1. This means that the Newton polygon either has ∞ as a slope, or has only one edge. The Hatcher-Thurston classification of incompressible surfaces in 2-bridge knot complements [**HT85**, Proposition 2] shows that actually ∞ cannot be a slope, in fact all boundary slopes are integers.

Thus the Newton polygon has only one edge. For a hyperbolic knot the Newton polygon has at least two distinct sides. Thus the knot is non-hyperbolic.

Since 2-bridge knots are alternating ([**BZ03**]) a theorem of Menasco [**Men84**] says that the knot can only be a torus knot. Since the bridge number of a torus knot $T(p, q)$ is at least $\min\{p, q\}$, the torus knot must be $T(2, p) = \mathfrak{b}(p, 1)$.

Note that for a torus knot $T(2, p)$ indeed $A(L, M) = LM^{2p} + 1$ ([**HS04**, **Zha04**]) having L -degree 1. □

THEOREM 3.2.5. *If p is prime then the A -polynomial of $\mathfrak{b}(p, q)$ is irreducible. Furthermore if $(p-1)/2$ is also prime and $q \neq 1$ then the L -degree of $A(L, M)$ is $(p-1)/2$.*

PROOF. The first part follows from Propositions 3.2.1, 3.2.2 and 3.2.3. We prove the second part.

First we claim that the y -degree of $\Phi_{(p,q)}(M, y)$ is $n = (p-1)/2$. Indeed, look at $\Phi_{(p,q)}(M, y) = \text{tr } w - \text{tr } w' + \cdots + (-1)^{n-1} \text{tr } w^{(n-1)} + (-1)^n$. Because the letter b appears n times in the word w , the entries of the matrix $\rho(w)$ have z -degrees, hence y -degrees, at most n . So the y -degree of $\Phi_{(p,q)}(M, y)$ is at most n . On the other hand Proposition 3.2.2 and the proof of Proposition 3.2.1 show that the y -degree is at least n , so the claim follows.

From the determinant description of resultant ([Lan93, p. 200]) it is clear that $\text{Res}(\Phi(M, y), \mathcal{L}(M, y) - L)$ has degree n in L . Since $A(L, M)$ is irreducible we have a positive integer k such that $A^k(M, L) = \text{Res}(\Phi(M, y), \mathcal{L}(M, y) - L)$. Thus the L -degree ℓ of $A(L, M)$ must be a factor of n . If n is prime then ℓ can only be 1 or n . If $\ell = 1$ then the knot is a torus knot and $q = 1$ according to Proposition 3.2.4. \square

CHAPTER 4

The colored Jones polynomial and Kashaev invariant

4.1. Introduction

For a knot K in \mathbb{R}^3 , the colored Jones polynomial $J'_K(N)$ is a Laurent polynomial, $J'_K(N) \in \mathcal{R} := \mathbb{Z}[q^{\pm 1}]$, see [Jon87, MM95]. Here N is a positive integer standing for the N -dimensional prime \mathfrak{sl}_2 -module. We use the unframed version and the normalization in which $J'_K(N) = 1$ when K is the unknot. The colored Jones polynomial $J'_K(N)$ is defined using the R -matrix of the quantized enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$.

Here we present the colored Jones polynomial as the inverse of the quantum determinant of an almost quantum matrix whose entries are in the q -Weyl algebra of q -operators acting on the polynomial rings, evaluated at the constant function 1. The proof is based on the quantum MacMahon Master theorem proved in [GLZ03]. Actually, it was an attempt to get a determinant formula for the colored Jones polynomial that led to the conjecture that eventually became the quantum MacMahon's Master theorem in [GLZ03].

We will then give an application to the case of the Kashaev invariant $\langle K \rangle_N := J'_K(N)|_{q=\exp 2\pi i/N}$. We show that a special evaluation of the determinant will give the Kashaev invariant. Our interpretation of the Kashaev invariant suggests that the natural generalization of the Kashaev invariant for other simple Lie algebra should be the quantum invariant of knots colored by the Verma module of highest weight $-\delta$, where δ is the half-sum of positive roots.

Finally we point out how the hyperbolic volume of the knot complement, through the theory of L^2 -torsion, has a determinant formula that looks strikingly similar to

the one of Kashaev invariants: In both we have non-commutative deformations of the Burau matrices, but in one case quantum determinant is used, in the other the Fuglede-Kadison determinant is used. This suggests an approach to the volume conjecture using quantum determinant as an approximation of the infinite-dimensional Fuglede-Kadison determinant.

4.1.1. A determinant formula for the colored Jones polynomial.

4.1.1.1. *Right-quantum matrices and quantum determinants.* A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is *right-quantum* if

$$ac = qca \quad (\text{q-commutation of the entries in a column})$$

$$bd = qdb \quad (\text{q-commutation of the entries in a column})$$

$$ad = da + qcb - q^{-1}bc \quad (\text{cross commutation relation}).$$

An $m \times m$ matrix is *right-quantum* if any 2×2 submatrix of it is right-quantum. The meaning is a right-quantum matrix preserves the structure of quantum m -spaces (see [Man88]). The product of 2 right-quantum matrices is a right-quantum matrix, provided that every entry of the first commutes with every entry of the second. The quantum determinant of any right-quantum $A = (a_{ij})$ is defined by

$$\det_q(A) := \sum_{\pi} (-q)^{\text{inv}(\pi)} a_{\pi 1,1} a_{\pi 2,2} \dots a_{\pi m,m}$$

where the sum ranges over all permutations of $\{1, \dots, m\}$, and $\text{inv}(\pi)$ denotes the number of inversions.

Note that in general $I - A$, where I is the identity matrix, is not right-quantum any more. We will define its determinant, using an analog of the expansion in the case

$q = 1$:

$$\widetilde{\det}_q(I - A) := 1 - C, \quad \text{where} \quad C := \sum_{\emptyset \neq J \subset \{1, 2, \dots, r\}} (-1)^{|J|-1} \det_q(A_J),$$

where A_J is the J by J submatrix of A , which is always right-quantum.

4.1.1.2. *Deformed Burau matrix.* On the polynomial ring $\mathcal{R}[x^{\pm 1}, y^{\pm 1}, u^{\pm 1}]$ act operators \hat{x}, τ_x and their inverses:

$$\hat{x}f(x, y, \dots) := xf(x, y, \dots), \quad \tau_x f(x, y, \dots) := f(qx, y, \dots).$$

It's easy to see that $\hat{x}\tau_x = q\tau_x\hat{x}$. For other variable, say y , there are similar operators \hat{y}, τ_y , each of which commutes with each of \hat{x}, τ_x . Let us define

$$(4.1.1) \quad a_+ = (\hat{u} - \hat{y}\tau_x^{-1})\tau_y^{-1}, \quad b_+ = \hat{u}^2, \quad c_+ = \hat{x}\tau_y^{-2}\tau_u^{-1},$$

$$(4.1.2) \quad a_- = (\tau_y - \hat{x}^{-1})\tau_x^{-1}\tau_u, \quad b_- = \hat{u}^2, \quad c_- = \hat{y}^{-1}\tau_x^{-1}\tau_u.$$

Then it is easy to check that the following matrices S_{\pm} are right-quantum.

$$S_+ := \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \quad S_- := \begin{pmatrix} 0 & c_- \\ b_- & a_- \end{pmatrix}$$

Suppose P is a polynomial in the operators $a_{\pm}, b_{\pm}, c_{\pm}$ with coefficients in $\mathcal{R} = \mathbb{Z}[q^{\pm 1}]$. Applying P to the constant function 1, then substituting u by 1 and x and y by z , one gets a polynomial $\mathcal{E}(P) \in \mathbb{Z}[q^{\pm 1}, z^{\pm 1}]$. Then it is readily seen that $\mathcal{E}(S_+)$ and $\mathcal{E}(S_-)$ are the transpose Burau matrix and its inverse:

$$\mathcal{E}(S_+) = \begin{pmatrix} 1 - z & 1 \\ z & 0 \end{pmatrix}, \quad \mathcal{E}(S_-) = \begin{pmatrix} 0 & z^{-1} \\ 1 & 1 - z^{-1} \end{pmatrix}.$$

4.1.1.3. *Determinant formula.* Let $\sigma_i, 1 \leq i \leq m - 1$, be the standard generators of the braid group on m strands, see for example [Bir74, Jon87]. For a sequence

$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k)$ of pairs $\gamma_j = (i_j, \varepsilon_j)$, where $1 \leq i_j \leq m - 1$ and $\varepsilon_j = \pm$, let $\beta = \beta(\gamma)$ be the braid

$$\beta := \sigma_{i_1}^{\varepsilon_1} \sigma_{i_2}^{\varepsilon_2} \dots \sigma_{i_k}^{\varepsilon_k}.$$

Here σ^\pm means $\sigma^{\pm 1}$. We will assume that the closure of β (see [Bir74]) is a *knot*, i.e. it has only one connected component. Recall that in the Burau representation of the braid $\beta(\gamma)$, we associate to each $\sigma_{i_j}^{\varepsilon_j}$ an $m \times m$ matrix which is the same as the identity matrix everywhere except for the 2×2 minor of rows $i_j, i_j + 1$ and columns $i_j, i_j + 1$, where we put the 2×2 Burau matrix if $\varepsilon_j = +$, or its inverse if $\varepsilon_j = -$. Let us do the same, only now the 2×2 Burau matrix and its inverse, for $\sigma_{i_j}^{\varepsilon_j}$, are replaced by $S_{+,j}$ and $S_{-,j}$. Here $S_{\pm,j}$ are the same as S_\pm with x, y, u replaced by x_j, y_j, u_j . For the precise definition see Section 4.2.2.2. The result is a right-quantum matrix $\rho(\gamma)$, whose entries are operators acting on $\mathcal{P}_k = \otimes_{j=1}^k \mathcal{R}[x_j^{\pm 1}, y_j^{\pm 1}, u_j^{\pm 1}]$. Note that $\rho(\gamma)$ might not be an invariant of the braid $\beta(\gamma)$. We can define $\mathcal{E}(P)$, where P is an operator acting on \mathcal{P}_k , as before: first apply P to the constant function 1, then replace all the u_j with 1, and all the x_j and y_j with z . Further, let $\mathcal{E}_N(P)$ be obtained from $\mathcal{E}(P)$ by the substitution $z \rightarrow q^{N-1}$.

Let $\rho'(\gamma)$ be obtained from $\rho(\gamma)$ by removing the first row and column. Let $w(\beta)$ denotes the writhe, $w(\beta) := \sum_j \varepsilon_j$. It's easy to show that when the closure of β is a knot, $w(\beta) - m + 1$ is always even.

THEOREM 4.1.1. *Suppose the closure in the standard way of the m -strand braid $\beta(\gamma)$ is a knot K .*

a). *For any positive integer N one has*

$$q^{(N-1)(w(\beta)-m+1)/2} \mathcal{E}_N \left(\frac{1}{\widetilde{\det}_q(I - q \rho'(\gamma))} \right) = J'_K(N).$$

b). *$\det \mathcal{E}(I - \rho'(\gamma))$ is equal to the Alexander polynomial of K .*

Part a) should be understood as follows. Suppose $\widetilde{\det}_q(I - \rho'(\gamma)) = 1 - C$, then when applying \mathcal{E}_N to

$$(4.1.3) \quad \frac{1}{1 - C} := \sum_{n=0}^{\infty} C^n,$$

only a finite number of terms are non-zero, hence the sum is well-defined, and is equal to the colored Jones polynomial. We would like to emphasize that here $N > 0$. If $N = 0$, when applying \mathcal{E}_N to the right hand side of (4.1.3), there might be infinitely many non-zero terms. From the theorem one can immediately get the Melvin-Morton conjecture, first proved by Bar-Natan and Garoufalidis [BNG96].

REMARK 4.1.2. Another determinant formula of the colored Jones polynomial using non-commutative variables was given in the independent work [GL04], also based on the quantum MacMahon Master theorem. The main difference is here our variables are explicit operators acting on polynomials ring. This sometimes helps since operators can be composed. Another difference is we derive our formula from the R -matrix, while [GL04] used cablings of the original Jones polynomial and graph theory. Our approach is a non-commutative analog of Rozansky's beautiful work [Roz98].

4.1.1.4. *An example.* To see an application of our formula let's calculate the colored Jones polynomial of the right-handed trefoil. In this case we need only 2 strands with $\beta = \sigma^3$. Thus $\rho(\gamma) = S_{+,1}S_{+,2}S_{+,3}$ is easy to calculate, and we get $\rho'(\gamma) = c_1a_2b_3$. Hence, with K being the right-handed trefoil,

$$(4.1.4) \quad \begin{aligned} J'_K(N) &= q^{N-1} \mathcal{E}_N \left(\frac{1}{1 - qc_1a_2b_3} \right) = q^{N-1} \sum_{n=0}^{\infty} \mathcal{E}_N(q^n c_1^n a_2^n b_3^n) \\ &= q^{N-1} \sum_{n=0}^{\infty} q^{nN} (1 - q^{N-1})(1 - q^{N-2}) \dots (1 - q^{N-n}). \end{aligned}$$

Note that the sum is always finite, since the term in the right hand side is 0 if $n \geq N$.

4.1.2. The Kashaev's invariant as the invariant of dimension 0. Kashaev [Kas97] used quantum dilogarithm to define a knot invariant $\langle K \rangle_N$, depending on a positive number N . Murakami and Murakami [MM01] showed that $\langle K \rangle_N = J'_K(N)|_{q=\exp(2\pi i/N)}$. The famous volume conjecture [Kas97, MM01] says that the growth rate of $\langle K \rangle_N$ is equal to the volume $V(K)$ (see definition below) of the knot complement:

$$\lim_{N \rightarrow \infty} \frac{\ln |\langle K \rangle_N|}{N} = \frac{\text{Vol}(K)}{2\pi}.$$

Working with varying N , i.e. working with varying \mathfrak{sl}_2 -modules might be difficult. Here we show that the values of $\langle K \rangle_N$ comes from just one \mathfrak{sl}_2 -module, the Verma module of highest weight -1 , and is a kind of analytic function in the following sense. Let us define the Habiro ring $\widehat{\mathbb{Z}[q]}$ by

$$\widehat{\mathbb{Z}[q]} := \varprojlim \mathbb{Z}[q]/((1-q)(1-q^2)\dots(1-q^n)).$$

Habiro [Hab02] called it the cyclotomic completion of $\mathbb{Z}[q]$. Formally, $\widehat{\mathbb{Z}[q]}$ is the set of all series of the form

$$f(q) = \sum_{n=0}^{\infty} f_n(q) (1-q)(1-q^2)\dots(1-q^n), \quad \text{where } f_n(q) \in \mathbb{Z}[q].$$

Suppose U is the set of roots of 1. If $\xi \in U$ then $(1-\xi)(1-\xi^2)\dots(1-\xi^n) = 0$ if n is big enough, hence one can define $f(\xi)$ for $f \in \widehat{\mathbb{Z}[q]}$. One can consider every $f \in \widehat{\mathbb{Z}[q]}$ as a function with domain U . Note that $f(\xi) \in \mathbb{Z}[\xi]$ is always an algebraic integer. It turns out $\widehat{\mathbb{Z}[q]}$ has remarkable properties, and plays an important role in quantum topology. First, each $f \in \widehat{\mathbb{Z}[q]}$ has a natural Taylor series at every point of U , and if two functions $f, g \in \widehat{\mathbb{Z}[q]}$ have the same Taylor series at a point in U , then $f = g$. A consequence is that $\widehat{\mathbb{Z}[q]}$ is an integral domain. Second, if $f = g$ at infinitely many roots of prime power orders, then $f = g$ (see [Hab02]). Hence one can consider $\widehat{\mathbb{Z}[q]}$

as a class of “analytic functions” with domain U . It was proved, by Habiro for \mathfrak{sl}_2 and by Habiro with Le for general simple Lie algebras, that quantum invariants of integral homology 3-spheres belong to $\widehat{\mathbb{Z}[q]}$ and thus have remarkable integrality properties. Here we show that the Kashaev invariant also belongs to $\widehat{\mathbb{Z}[q]}$:

THEOREM 4.1.3. *a). $q^{(m-w(\beta)-1)/2} \mathcal{E}_0 \left(\frac{1}{\widetilde{\det}_q(I - q\rho'(\gamma))} \right)$ belongs to $\widehat{\mathbb{Z}[q]}$ and is an invariant of the knot K obtained by closing $\beta(\gamma)$.*

b). Kashaev’s invariant is equal to

$$(4.1.5) \quad \langle K \rangle_N = q^{(m-w(\beta)-1)/2} \mathcal{E}_0 \left(\frac{1}{\widetilde{\det}_q(I - q\rho'(\gamma))} \right) \Big|_{q=\exp(2\pi i/N)}.$$

For example, when K is the left-handed trefoil, from (4.1.4), with $q \rightarrow q^{-1}$, we have

$$\langle K \rangle_N = q \sum_{n=0}^{\infty} (1 - q)(1 - q^2) \dots (1 - q^n),$$

where $q = \exp(2\pi i/N)$. The function given by the infinite sum on the right hand side was first written down by M. Kontsevich, and its asymptotics was completely determined by Zagier [Zag01]. We see that it has a nice geometric interpretation: It is the Kashaev invariant of the trefoil.

4.1.3. Hyperbolic volume and L^2 -torsion. It is known that by cutting the knot complement $S^3 \setminus K$ along some embedded tori one gets connected components which are either Seifert-fibered or hyperbolic. Let $\text{Vol}(K)$ be the sum of the hyperbolic volume of the hyperbolic pieces, ignoring the Seifert-fibered components. It’s known that $\text{Vol}(K)$ is proportional to the Gromov norm [BP92], and can be calculated using L^2 -torsion as follows. Let the knot K again be the closure of the braid β . The fundamental group of the knot complement has a presentation:

$$\pi_1 = \langle z_1, \dots, z_m \mid r_1, \dots, r_m \rangle,$$

where $r_i = \beta(z_i)z_i^{-1}$, with β considered as an automorphism of the free group on m generators z_1, \dots, z_m .

Let $Ja = \left(\frac{\partial r_i}{\partial z_j}\right)$ be the Jacobian matrix with entries in $\mathbb{Z}[\pi_1]$, where $\frac{\partial r_i}{\partial z_j}$ is the Fox derivative. For a matrix with entries in $\mathbb{Z}[\pi_1]$, one can define its Fuglede-Kadison determinant (see [Lüc02]), denoted by \det_{π_1} . A deep theorem of Luck and Schick [Lüc02] says that

$$\text{Vol}(K) = 6\pi \ln(\det_{\pi_1}(Ja')),$$

where Ja' is obtained from Ja by removing the first row and column. It's easy to see that

$$Ja = \psi(\beta) - I, \quad \text{where} \quad \psi(\beta) = \left(\frac{\partial(\beta(z_i))}{\partial z_j}\right).$$

A simple property of Fuglede-Kadison determinant is that $\det_{\pi_1}(A) = \det_{\pi_1}(-A)$. Hence we have

PROPOSITION 4.1.4. *Let $\psi'(\beta)$ be obtained from $\psi(\beta)$ by removing the first row and column. Then*

$$(4.1.6) \quad \exp\left(-\frac{\text{Vol}(K)}{6\pi}\right) = \frac{1}{\det_{\pi_1}(I - \psi'(\beta))}.$$

Note that under the abelianization map $ab : \mathbb{Z}[\pi_1] \rightarrow \mathbb{Z}[\mathbb{Z}]$, the matrix $\psi(\beta)$ becomes the Burau representation of β . Hence both $\psi(\beta)$ and $\rho(\beta(\gamma))$ are two different kinds of quantization of the Burau representation. We hope that the similarity between (4.1.6) and (4.1.5) will help to solve the volume conjecture. One needs to relate the Fuglede-Kadison determinant \det_{π_1} to the quantum determinant.

Also note that the abelianized version of the the right hand side of (4.1.6), i.e. $\det_{\mathbb{Z}}(I - ab(\psi'(\beta)))$, is equal to the Mahler measure of the Alexander polynomial (see [Lüc02]). This partially explains some similarity between the Mahler measure and the hyperbolic volume of a knot, as observed in [SW04].

4.1.4. Plan of the chapter. In section 4.2 we prove Theorem 4.1.1. Section 4.3.1 contains a proof of Theorem 2 and a discussion about generalization to other Lie algebra of the Kashaev invariants.

4.2. Proof of Theorem 4.1.1

In subsection 4.2.1 we recall the definition of the colored Jones polynomial using R -matrix. We will follow Rozansky [Roz98] to twist the R -matrix so that it has a “nice” form. Then in the subsequent subsections we show how the twisted R -matrix can be obtained from the deformed Burau matrix, giving a proof of Theorem 4.1.1.

We will use the variable $v^{1/2}$ such that $v^2 = q$. Note that our q is equal to q^2 in [Jan96]. Recall that $\mathcal{R} = \mathbb{Z}[q^{\pm 1}]$, which is a subring of the field $\tilde{\mathcal{R}} := \mathbb{C}(v^{\pm 1/2})$. We will use the following standard notations.

$$\begin{aligned}
 [n] &:= \frac{v^n - v^{-n}}{v - v^{-1}}, & [n]! &:= \prod_{i=1}^n [i], & \begin{bmatrix} n \\ l \end{bmatrix} &:= \prod_{i=1}^l \frac{[n-i+1]}{[l-i+1]}, \\
 (n)_q &:= \frac{1 - q^{-n}}{1 - q^{-1}}, & \binom{n}{l}_q &:= \prod_{i=1}^l \frac{(n-i+1)_q}{(l-i+1)_q}, & (1-x)_q^d &:= \prod_{i=0}^{d-1} (1-xq^i).
 \end{aligned}$$

4.2.1. The colored Jones polynomial through R -matrix.

4.2.1.1. *The quantized enveloping algebra $U_v(\mathfrak{sl}_2)$.* Let \mathcal{U} be the algebra over the field $\tilde{\mathcal{R}} = \mathbb{C}(v^{\pm 1/2})$ generated by $K^{\pm 1/2}, E, F$, subject to the relation

$$K^{1/2}K^{-1/2} = 1, \quad K^{1/2}E = vEK^{1/2}, \quad K^{1/2}F = v^{-1}FK^{1/2}, \quad EF - FE = \frac{K - K^{-1}}{v - v^{-1}}.$$

Then \mathcal{U} is a Hopf algebra with coproduct:

$$\Delta(K^{1/2}) = K^{1/2} \otimes K^{1/2}, \quad \Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F.$$

Here we follow the definition of Jantzen's book [Jan96], only we add the square root $K^{1/2}$ for convenience. Note that $V \otimes W$ has a natural \mathcal{U} -module structure whenever V, W have, due to the co-algebra structure.

4.2.1.2. *The quasi- R -matrix and braiding.* The quasi- R -matrix Θ is an element of some completion of \mathcal{U} :

$$\Theta := \sum_{n=0}^{\infty} (-1)^n v^{-n(n-1)/2} \frac{(v - v^{-1})^n}{[n]!} F^n \otimes E^n.$$

An \mathcal{U} -module V is *E -locally-finite* if for every $u \in V$ there is n such that $E^n u = 0$. If V and W are E -locally-finite, then for every $u \otimes w \in V \otimes W$, there are only a finite number of terms in the sum of Θ that do not annihilate $u \otimes w$, hence we can define Θ as an $\tilde{\mathcal{R}}$ -linear operator acting on $V \otimes W$. The inverse of Θ is given by

$$\Theta^{-1} := \sum_{n=0}^{\infty} v^{n(n-1)/2} \frac{(v - v^{-1})^n}{[n]!} F^n \otimes E^n.$$

An element u in an \mathcal{U} -module is said to have weight l if $Ku = v^l u$. We will consider only \mathcal{U} -modules that are spanned by weight vectors. For such modules V and W we define the diagonal operator D by

$$D(u \otimes w) = v^{-kl/2} u \otimes w,$$

where u has weight k , w has weight l . The braiding $\mathfrak{b} : V \otimes W \rightarrow W \otimes V$ is defined by

$$\mathfrak{b}(u \otimes w) := \Theta(D(w \otimes u)).$$

It's known that \mathfrak{b} commutes with the action of \mathcal{U} , is invertible, and satisfies the braid relation: Suppose V is an E -locally-finite \mathcal{U} -module. Let $\mathfrak{b}_{12} := \mathfrak{b} \otimes \text{id}$ and $\mathfrak{b}_{23} := \text{id} \otimes \mathfrak{b}$ be the operators acting on $V \otimes V \otimes V$. Then

$$\mathfrak{b}_{12} \mathfrak{b}_{23} \mathfrak{b}_{12} = \mathfrak{b}_{23} \mathfrak{b}_{12} \mathfrak{b}_{23}.$$

One can define a representation of the braid group on m strands into the group of linear operators acting on $V^{\otimes m}$ by putting

$$\tau(\sigma_i) = \text{id}^{\otimes(i-1)} \otimes \mathfrak{b} \otimes \text{id}^{\otimes(m-i-1)},$$

i.e. σ_i acts trivially on all components, except for the i -th and $(i+1)$ -st where it acts as \mathfrak{b} .

4.2.1.3. *A modification of Verma module V_N .* For an integer N , not necessarily positive, let V_N be the $\tilde{\mathcal{R}}$ -vector space freely spanned by $e_i, i \in \mathbb{Z}_{\geq 0}$. The following can be readily checked.

PROPOSITION 4.2.1. *The space V_N has a structure of an E -locally-finite \mathcal{U} -module given by*

$$\begin{aligned} Ke_i &= v^{N-1-2i} e_i \\ Ee_i &= (i)_{q^{-1}} e_{i-1} \\ Fe_i &= v^i [N-1-i] e_{i+1} = \frac{v^{1-N}}{v-v^{-1}} (q^{N-1} - q^i) e_{i+1}. \end{aligned}$$

For $N > 0$ let W_N be the $\tilde{\mathcal{R}}$ -subspace of V_N spanned by $e_i, 0 \leq i \leq N-1$. It's is easy to see that W_N is a simple \mathcal{U} -submodule of V_N . Every simple finite dimensional \mathcal{U} -module is isomorphic to one of W_N .

REMARK 4.2.2. The traditional basis $e'_i := F^i(e_0)/[i]!$ is related to the basis e_i by

$$\begin{bmatrix} N-1 \\ i \end{bmatrix} e_i = v^{-i(i-1)/2} e'_i.$$

4.2.1.4. *The colored Jones polynomial.* If the closure of the m -strand braid β is the knot K , then the colored Jones polynomial $J_K(N)$ can be defined as the quantum

trace of $\tau(\beta)$ on $(W_N)^{\otimes m}$:

$$J_K(N) = v^{w(\beta)\frac{N^2-1}{2}} \operatorname{tr}_q(\tau(\beta), (W_N)^{\otimes m}) := v^{w(\beta)\frac{N^2-1}{2}} \operatorname{tr}(\tau(\beta)K^{-1}, (W_N)^{\otimes m}).$$

Here $w(\beta) := \sum_j \varepsilon_j 1$ is the writhe of β . The factor $v^{w(\beta)\frac{N^2-1}{2}}$ will make $J_K(N)$ not depending on the framing. If K is the unknot then $J_K(N) = [N]$. The normalized version $J'_K(N) := J_K(N)/[N]$ can be calculated using the partial trace as follows. Recall that $\tau(\beta)$ acts on $(W_N)^{\otimes m}$. Taking the quantum trace of $\tau(\beta)$ in only the $m-1$ last components, we get an operator acting on the first W_N , which is known to be a scalar times the identity operator, with the scalar being exactly $J'_K(N)$. This can be written in the formula form as follows. Let $p_0 : (V_N)^{\otimes m} \rightarrow (V_N)^{\otimes m}$ be the projection onto $e_0 \otimes (V_N)^{\otimes(m-1)}$, i.e.

$$p_0(e_{n_1} \otimes e_{n_2} \otimes \cdots \otimes e_{n_m}) = \delta_{0, i_1} e_{n_2} \otimes \cdots \otimes e_{n_m}.$$

Then p_0 also restricts to a projection from $(W_N)^{\otimes m}$ onto $e_0 \otimes (W_N)^{\otimes(m-1)}$, and

$$(4.2.1) \quad J'_K(N) = v^{w(\beta)\frac{N^2-1}{2}} \operatorname{tr}(p_0(\tau(\beta)K^{-1}), e_0 \otimes (W_N)^{\otimes(m-1)}).$$

4.2.1.5. *Twisting the braiding.* It's straightforward to calculate the action of the braiding \mathfrak{b} on $V_N \otimes V_N$, using the basis $e_{n_1} \otimes e_{n_2}$, $n_1, n_2 \in \mathbb{Z}_{\geq 0}$. However to get a better, more convenient form we will follow Rozansky [**Roz98**] to use the twisted braiding

$$\check{\mathfrak{b}} := Q^{-1} \mathfrak{b} Q \quad \text{where} \quad Q = \operatorname{id} \otimes K^{(1-N)/2}.$$

Then direct calculation shows that on $V_N \otimes V_N$ the action of the twisted braiding $\check{\mathfrak{b}}_{\pm}$ are given by

$$\check{\mathfrak{b}}_{\pm}(e_{n_1} \otimes e_{n_2}) = \sum_{l=0}^{\max n_1, n_2} \check{\mathfrak{b}}_{\pm}(n_1, n_2; l) (e_{n_2 \pm l} \otimes e_{n_1 \mp l}),$$

where, with $z = q^{N-1}$,

$$(4.2.2) \quad (\check{\mathfrak{b}}_+)(n_1, n_2; l) = q^{-\frac{(N-1)^2}{4}} \binom{n_1}{l}_{q^{-1}} q^{n_2(l-n_1)} z^{n_2} (1 - zq^{-n_2})_{q^{-1}}^l$$

$$(4.2.3) \quad (\check{\mathfrak{b}}_-)(n_1, n_2; l) = q^{\frac{(N-1)^2}{4}} \binom{n_2}{l}_q q^{n_1(n_2-l)} z^{-n_1} (1 - z^{-1}q^{n_1})_q^l.$$

Note that our formulas differ from those in [Roz98] by $q \rightarrow q^{-1}$, since we derived our formula directly from the quantized enveloping algebra that differs from the one implicitly used by Rozansky. (The co-products are opposite; “implicitly” since Rozansky never used quantized enveloping algebra, but just took the formula of the R -matrix from [KM91]).

To justify the use of the twisted braiding we argue as follows. First note that \mathfrak{b}_\pm commutes with $K^{1/2}$, the action of which on $V_N \otimes V_N$ is given by $\Delta(K^{1/2}) = K^{1/2} \otimes K^{1/2}$. Thus $K^l \otimes K^l$ commutes with \mathfrak{b}_\pm for every half-integer l . Hence

$$(4.2.4) \quad (Q')^{-1} \mathfrak{b}_\pm Q' = Q^{-1} \mathfrak{b}_\pm Q = \check{\mathfrak{b}}_\pm$$

if

$$Q' = Q (K^{i(1-N)/2} \otimes K^{i(1-N)/2}) = K^{i(1-N)/2} \otimes K^{(i+1)(1-N)/2}.$$

Let us define the operator Q_m acting on $(W_N)^{\otimes m}$ by

$$Q_m := K^{(1-N)/2} \otimes K^{2(1-N)/2} \otimes \dots \otimes K^{m(1-N)/2}$$

and let

$$\check{\tau}(\beta) = Q_m^{-1} \tau(\beta) Q_m.$$

Then $\check{\tau}$ is also a representation of the braid group. Since the action of K^{-1} on $(W_N)^{\otimes m}$ commutes with the action of Q_m , one sees that in the formula (4.2.1) we can use $\check{\tau}(\beta)$

instead of $\tau(\beta)$:

$$(4.2.5) \quad J'_K(N) = v^{w(\beta)\frac{N^2-1}{2}} \operatorname{tr} (p_0(\check{\tau}(\beta) K^{-1}), e_0 \otimes (W_N)^{\otimes(m-1)}).$$

Suppose $\beta = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_k}^{\varepsilon_k}$. Then $\check{\tau}(\beta) = \check{\tau}(\sigma_{i_1})^{\varepsilon_1} \dots \check{\tau}(\sigma_{i_k})^{\varepsilon_k}$. Let us calculate $\check{\tau}(\sigma_i)$:

$$\begin{aligned} \check{\tau}(\sigma_i^{\pm 1}) &= Q_m^{-1} \tau(\sigma_i) Q_m \\ &= Q_m^{-1} (\operatorname{id}^{\otimes(i-1)} \otimes \mathfrak{b}_{\pm} \otimes \operatorname{id}^{\otimes(m-i-1)}) Q_m \\ &= \operatorname{id}^{\otimes(i-1)} \otimes ((K^{i(1-N)/2} \otimes K^{(i+1)(1-N)/2})^{-1} \mathfrak{b}_{\pm} (K^{i(1-N)/2} \otimes K^{(i+1)(1-N)/2})) \otimes \\ &\quad \otimes \operatorname{id}^{\otimes(m-i-1)} \\ &= \operatorname{id}^{\otimes(i-1)} \otimes \check{\mathfrak{b}}_{\pm} \otimes \operatorname{id}^{\otimes(m-i-1)} \quad \text{by (4.2.4)}. \end{aligned}$$

This means in the definition of $\check{\tau}$ one just use $\check{\mathfrak{b}}_{\pm}$ instead of \mathfrak{b}_{\pm} , and then $\check{\tau}$ is obtained from τ by the global twist Q_m .

4.2.1.6. *From W_N to V_N .* So far we take the trace using the finite dimensional module W_N . For the infinite dimensional V_N we define the trace of an operator if only a finite number of diagonal entries are nonzero. The following was observed in [Ro98].

LEMMA 4.2.3. *Suppose the closure of the braid β is a knot, then*

$$\begin{aligned} J'_K(N) &= v^{w(\beta)\frac{N^2-1}{2}} \operatorname{tr} (p_0(\check{\tau}(\beta) K^{-1}), e_0 \otimes (W_N)^{\otimes(m-1)}) \\ &= v^{w(\beta)\frac{N^2-1}{2}} \operatorname{tr} (p_0(\check{\tau}(\beta) K^{-1}), e_0 \otimes (V_N)^{\otimes(m-1)}). \end{aligned}$$

PROOF. One important observation is that if $n < N$, and $n+l \geq N$, then $F^l e_n = 0$. Hence $\check{\mathfrak{b}}_{\pm}(e_{n_1} \otimes e_{n_2})$ is a linear combination of $e_{m_1} \otimes e_{m_2}$ (with $m_1 + m_2 = n_1 + n_2$), and if $n_1 < N$ then $m_2 < N$, or if $n_2 < N$ then $m_1 < N$.

Let $(\tau(\beta)K^{-1})_{n_1, n_2, \dots, n_m}^{s_1, s_2, \dots, s_m}$ be the matrix of $\tau(\beta)K^{-1}$ with respect to the basis $e_{n_1} \otimes e_{n_2} \otimes \dots \otimes e_{n_m}$ in $(V_N)^{\otimes m}$. Note that K^{-1} acts diagonally in this basis. The above observation shows that if $n_i < N$ then $s_{\bar{\beta}(i)} < N$ for the matrix entry $(\tau(\beta)K^{-1})_{n_1, n_2, \dots, n_m}^{s_1, s_2, \dots, s_m}$ not to be 0, where $\bar{\beta}$ is the permutation corresponding to β . To take the trace we only have to concern with the case $s_i = n_i$. We have already had $n_1 = 0$, which is less than N . Thus we must have $n_j < N$ for $j = 1, \bar{\beta}(1), (\bar{\beta})^2(1) \dots$. The fact that the closure of β is a knot implies that $\{(\bar{\beta})^l(1), 1 \leq l \leq m\}$ is the whole set $\{1, 2, \dots, m\}$. Hence taking the trace over $e_0 \otimes (V_N)^{\otimes(m-1)}$ is the same as over $e_0 \otimes (W_N)^{\otimes(m-1)}$. \square

4.2.2. Algebra of the deformed Burau matrix.

4.2.2.1. *Algebra \mathcal{A}_ε .* Let us define

$$\mathcal{A}_+ := \mathcal{R}\langle a_+, b_+, c_+ \rangle / (a_+ b_+ = b_+ a_+, a_+ c_+ = q c_+ a_+, b_+ c_+ = q^2 c_+ b_+).$$

$$\mathcal{A}_- := \mathcal{R}\langle a_-, b_-, c_- \rangle / (a_- b_- = q^2 b_- a_-, c_- a_- = q a_- c_-, c_- b_- = q^2 b_- c_-).$$

It is easy to check that the a_\pm, b_\pm, c_\pm of section 4.1.1.2 satisfy the commutation relations of the algebras \mathcal{A}_\pm .

For a sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$, where each ε_j is either $+$ or $-$, let $\mathcal{A}_\varepsilon = \mathcal{A}_{\varepsilon_1} \otimes \mathcal{A}_{\varepsilon_2} \otimes \dots \otimes \mathcal{A}_{\varepsilon_k}$. We can consider \mathcal{A}_ε as the algebra over \mathcal{R} freely generated by a_j, b_j, c_j subject to the commutation relations: if $i \neq j$ then each of a_i, b_i, c_i commutes with each of a_j, b_j, c_j , if $\varepsilon_j = +$ then the commutations among a_j, b_j, c_j are the same as those of a_+, b_+, c_+ , and if $\varepsilon_j = -$ then the commutations among a_j, b_j, c_j are the same as those of a_-, b_-, c_- . Note that the algebra \mathcal{A}_ε is a generalized quantum space in the sense that for any a, b among the generators, one has the almost q -commutation relation $ab = q^l ba$, for some integer l .

Replacing $x, y, u, a_\pm, b_\pm, c_\pm$ with respectively $x_j, y_j, u_j, a_j, b_j, c_j$ in (4.1.1) if $\varepsilon_j = +$, or in (4.1.2) if $\varepsilon_j = -$, we identify a_j, b_j, c_j with operators acting on $\mathcal{R}[x_j^{\pm 1}, y_j^{\pm 1}, u_j^{\pm 1}]$. We assume that a_j, b_j, c_j leave alone x_i, y_i, u_i if $i \neq j$. Thus \mathcal{A}_ε acts on the algebra

\mathcal{P}_k of Laurent polynomials in $x_j, y_j, u_j, 1 \leq j \leq k$ with coefficients in \mathcal{R} . The map $\mathcal{E} : \mathcal{A}_\varepsilon \rightarrow \mathcal{R}[z^{\pm 1}]$ is defined as in section 4.1.1.2.

LEMMA 4.2.4. a). If $f, g \in \mathcal{A}_\varepsilon$ are separate, i.e. f contains only a_j, b_j, c_j with $j \leq r$ and g contains only a_l, b_l, c_l with $r < l$ (for some r), then $\mathcal{E}(fg) = \mathcal{E}(f)\mathcal{E}(g)$.

b). One has

$$(4.2.6) \quad \mathcal{E}(b_+^s c_+^r a_+^d) = q^{-rd} z^r (1 - zq^{-r})_q^d$$

$$(4.2.7) \quad \mathcal{E}(b_-^s c_-^r a_-^d) = z^{-r} (1 - z^{-1}q^r)_q^d$$

PROOF. a) follows directly from the definition. b) follows from an easy induction. \square

4.2.2.2. *Definition of $\rho(\gamma)$.* Let us give here the precise definition of $\rho(\gamma)$, for $\gamma = ((i_1, \varepsilon_1), \dots, (i_k, \varepsilon_k))$. Recall that β is the braid

$$\beta = \beta(\gamma) := \sigma_{i_1}^{\varepsilon_1} \sigma_{i_2}^{\varepsilon_2} \dots \sigma_{i_k}^{\varepsilon_k}.$$

If $\varepsilon_j = +$ (resp. $\varepsilon_j = -$), let S_j be the matrix S_+ (resp. S_-) with a_+, b_+, c_+ (resp. a_-, b_-, c_-) replaced by a_j, b_j, c_j . For the j -th factor $\sigma_{i_j}^{\varepsilon_j}$ let us define an $m \times m$ right-quantum matrix A_j by the block sum, just like in the Burau representation, only the non-trivial 2×2 block now is S_j instead of the Burau matrix:

$$A_j := I_{i_j-1} \oplus S_j \oplus I_{m-i_j-1}.$$

Here I_l is the identity $l \times l$ matrix.

Let $\rho(\gamma) := A_1 A_2 \dots A_k$. Then $\rho(\gamma)$ is an $m \times m$ right-quantum matrix with entries polynomials in a_j, b_j, c_j .

4.2.3. Quantum MacMahon Master Theorem.

4.2.3.1. *Co-actions of right-quantum matrices on the quantum space.* The quantum plane $\mathbb{C}_q[z_1, z_2, \dots, z_m]$, considered as the space of q -polynomial in the variables z_1, \dots, z_m , is defined as

$$\mathbb{C}_q[z_1, z_2, \dots, z_m] := \tilde{\mathcal{R}}\langle z_1, \dots, z_m \rangle / (z_i z_j = q z_j z_i \text{ if } i < j).$$

REMARK 4.2.5. Our definitions of quantum spaces, quantum matrices ... differ from the one in [GL03, Kas95] by the involution $q \rightarrow q^{-1}$, but agree with the ones in Jantzen's book [Jan96].

If $A = (a_{ij})_{i,j=1}^m$ is right-quantum and all a_{ij} 's commute with all z_1, \dots, z_m , then it is known that the $Z_i := \sum_j a_{ij} z_j$, i.e.

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \dots \\ Z_m \end{pmatrix} = A \begin{pmatrix} z_1 \\ z_2 \\ \dots \\ z_m \end{pmatrix},$$

also satisfy $Z_i Z_j = q Z_j Z_i$ if $i < j$. Let $\mathcal{W} = \mathcal{W}(A)$ be the algebra generated by $a_{ij}, 1 \leq i, j \leq m$, subject to the commutation relations of a_{ij} . Then we have an algebra homomorphism:

$$\Phi_A : \mathbb{C}_q[z_1, \dots, z_m] \rightarrow \mathcal{W} \otimes \mathbb{C}_q[z_1, \dots, z_m]$$

defined by $\Phi_A(z_i) = Z_i$. Informally, one could look at Φ_A as the degree-preserving algebra homomorphism on the q -polynomial ring $\mathbb{C}_q[z_1, z_2, \dots, z_m]$ defined by matrix A . Here we assume that the degree of each z_i is 1, and the degree of each a_{ij} is 0.

We will consider the case $A = \rho(\gamma)$, and in particular $A = \check{\mathfrak{b}}_{\pm}$. In this case we define $\mathcal{E}_N(\Phi_A) := (\mathcal{E}_N \otimes id) \circ \Phi_A$, which is a linear operator acting on $\mathbb{C}_q[z_1, \dots, z_m]$, not necessarily an algebra homomorphism.

4.2.3.2. *Quantum MacMahon Master theorem.* Let $\mathbb{C}_q[z_1, \dots, z_m]^{(n)}$ be the part of total degree n in $\mathbb{C}_q[z_1, \dots, z_m]$. Since Φ_A preserves the total degree, it restricts to a linear map: $\Phi_A : \mathbb{C}_q[z_1, \dots, z_m]^{(n)} \rightarrow \mathcal{W} \otimes \mathbb{C}_q[z_1, \dots, z_m]^{(n)}$. Let us define the trace by

$$\mathrm{tr} \left(\Phi_A, \mathbb{C}_q[z_1, \dots, z_m]^{(n)} \right) = \sum_{n_1 + \dots + n_m = n} (\Phi_A)_{n_1, \dots, n_m}^{n_1, \dots, n_m},$$

where $(\Phi_A)_{n_1, \dots, n_m}^{n_1, \dots, n_m}$ is the coefficients of $z_1^{n_1} \dots z_m^{n_m}$ in $Z_1^{n_1} \dots Z_m^{n_m}$. One could consider $\mathrm{tr} \left(\Phi_A, \mathbb{C}_q[z_1, \dots, z_m]^{(n)} \right)$ as the trace of Φ_A acting on the part of total degree n . The quantum MacMahon's Master theorem, proved in [GL03] says that

$$\frac{1}{\widetilde{\det}_q(I - A)} = \mathrm{tr} \left(\Phi_A, \mathbb{C}_q[z_1, \dots, z_m] \right) := \sum_{n=0}^{\infty} \mathrm{tr} \left(\Phi_A, \mathbb{C}_q[z_1, \dots, z_m]^{(n)} \right).$$

It's the q -analog of the identity

$$\frac{1}{\det(I - C)} = \sum_{n=0}^{\infty} \mathrm{tr}(S^n C),$$

where C is a linear operator acting on a finite dimensional \mathbb{C} -space V and $S^n C$ is the action of C on the n -th symmetric power of V .

4.2.4. From deformed Burau matrices S_{\pm} to R -matrices $\check{\mathfrak{b}}_{\pm}$. Let $\mathcal{F}_m : (V_N)^{\otimes m} \rightarrow \mathbb{C}_q[z_1, \dots, z_m]$ be the $\tilde{\mathcal{R}}$ -linear isomorphism defined by $\mathcal{F}(e_{n_1} \otimes \dots \otimes e_{n_m}) := z_1^{n_1} \dots z_m^{n_m}$. The following is important to us.

PROPOSITION 4.2.6. *a). Under the isomorphism \mathcal{F}_2 , the twisted braiding matrices $\check{\mathfrak{b}}_{\pm}$ acting on $V_N \otimes V_N$ map to $v^{\mp(N-1)^2/2} \mathcal{E}_N(S_{\pm})$, i.e.*

$$\check{\mathfrak{b}}_{\pm} = v^{\mp(N-1)^2/2} \mathcal{F}_2^{-1} \mathcal{E}_N(\Phi_{S_{\pm}}) \mathcal{F}_2.$$

b). Under the isomorphism \mathcal{F}_m , the linear automorphism $\check{\tau}(\beta(\gamma))$ of $(V_N)^{\otimes m}$ maps to $v^{\mp w(\beta)(N-1)^2/2} \mathcal{E}_N(\Phi_{\rho(\gamma)})$.

PROOF. a). Suppose for 2 variables X, Y we have $YX = qXY$, then Gauss's q-binomial formula [Kas95] says that

$$(X + Y)^n = \sum_{l=0}^n \binom{n}{l}_q X^l Y^{n-l}.$$

Let us first consider the case of S_+ . Then $\Phi_{S_+}(z_1) = a_+ z_1 + b_+ z_2$, and $\Phi_{S_-}(z_2) = c_+ z_1$. Note that $(b_+ z_2)(a_+ z_1) = q^{-1}(a_+ z_1)b_+(z_2)$, hence using the Gauss binomial formula we have

$$\begin{aligned} \Phi_{S_+}(z_1^{n_1} z_2^{n_2}) &= (a_+ z_1 + b_+ z_2)^{n_1} (c_+ z_1)^{n_2} \\ &= \sum_{l=0}^{n_1} \binom{n_1}{l}_{q^{-1}} (a_+ z_1)^l (b_+ z_2)^{n_1-l} (c_+ z_1)^{n_2} \\ &= \sum_{l=0}^{n_1} \binom{n_1}{l}_{q^{-1}} q^{n_2(n_1-l)} a_+^l b_+^{n_1-l} c_+^{n_2} (z_1)^{n_2+l} (z_2)^{n_1-l} \end{aligned}$$

Using formulas (4.2.2) and (4.2.6) one sees that

$$\check{\mathbf{b}}_+ = v^{-(N-1)^2/2} \mathcal{F}_2^{-1} \mathcal{E}_N(\Phi_{S_+}) \mathcal{F}_2.$$

The proof for S_- is quite similar, using formulas (4.2.3) and (4.2.7).

b). Because the variables x_j, y_j, u_j are separated, we have that

$$\mathcal{E}(\rho(\gamma)) = \mathcal{E}(\rho(\sigma_{i_1}^{\varepsilon_1})) \dots \mathcal{E}(\rho(\sigma_{i_k}^{\varepsilon_k})),$$

and the statement follows from part a). □

4.2.4.1. Under the isomorphism \mathcal{F}_m , the projection $p_0 : (V_N)^{\otimes m} \rightarrow (V_N)^{\otimes m}$ maps to the projection, also denoted by p_0 , of $\mathbb{C}_q[z_1, z_2, \dots, z_m]$, which can be defined as

$$p_0(z_1^{n_1} z_2^{n_2} \dots z_m^{n_m}) = \delta_{0, n_1} z_2^{n_2} \dots z_m^{n_m}.$$

Note that the kernel of p_0 is the ideal generated by z_1 .

- LEMMA 4.2.7. a). For every $u \in \mathbb{C}_q[z_2, \dots, z_m]$, $p_0(\Phi_{\rho(\beta)}(u)) = \Phi_{\rho'(\beta)}(u)$.
 b). The operators p_0 and \mathcal{E}_N commute: $p_0(\mathcal{E}_N(\Phi_{\rho(\beta)}(u))) = \mathcal{E}_N(p_0(\Phi_{\rho(\beta)}(u)))$.

PROOF. a). Recall that $\rho'(\gamma)$ is obtained from $\rho(\gamma)$ by removing the first row and column. Suppose $u \in \mathbb{C}_q[z_2, \dots, z_m]$, then $\Phi_{\rho(\beta)}(u) - \Phi_{\rho'(\beta)}(u)$ is divisible by z_1 , and hence annihilated by p_0 .

b). follows trivially from the definition. \square

The following is trivial.

LEMMA 4.2.8. Under \mathcal{F}_m , the action of K^{-1} on $\mathbb{C}_q[z_2, \dots, z_m]^{(n)}$ is the scalar operator, with scalar $v^{(m-1)(1-N)+2n} = v^{(m-1)(1-N)}q^n$.

4.2.5. Proof of Theorem 4.1.1.

$$\begin{aligned}
 J'_K(N) &= v^{\frac{w(\beta)(N-1)^2}{2}} \operatorname{tr}(p_0(\check{\tau}(\beta)K^{-1}), e_0 \otimes (W_N)^{\otimes(m-1)}) \quad \text{by Lemma 4.2.3} \\
 &= v^{w(\beta)(N-1)} \operatorname{tr}(p_0(\mathcal{E}_N(\Phi_{\rho(\gamma)})K^{-1}), \mathbb{C}_q[z_2, \dots, z_m]) \\
 &\quad \text{under } \mathcal{F}_m, \text{ by Proposition 4.2.6} \\
 &= v^{w(\beta)(N-1)} \operatorname{tr}(\mathcal{E}_N(\Phi_{\rho'(\gamma)})K^{-1}, \mathbb{C}_q[z_2, \dots, z_m]) \quad \text{by Lemma 4.2.7} \\
 &= v^{w(\beta)(N-1)} \sum_{n=0}^{\infty} \operatorname{tr}(\mathcal{E}_N(\rho'(\gamma))K^{-1}, \mathbb{C}_q[z_2, \dots, z_m]^{(n)}) \\
 &= v^{(w(\beta)-m+1)(N-1)} \sum_{n=0}^{\infty} q^n \operatorname{tr}(\mathcal{E}_N(\Phi_{\rho'(\gamma)}), \mathbb{C}_q[z_2, \dots, z_m]^{(n)}) \quad \text{by Lemma 4.2.8} \\
 &= v^{(w(\beta)-m+1)(N-1)} \sum_{n=0}^{\infty} \operatorname{tr}(\mathcal{E}_N(\Phi_{q\rho'(\gamma)}), \mathbb{C}_q[z_2, \dots, z_m]^{(n)}) \\
 &= v^{(w(\beta)-m+1)(N-1)} \mathcal{E}_N \sum_{n=0}^{\infty} \operatorname{tr}(\Phi_{q\rho'(\gamma)}, \mathbb{C}_q[z_2, \dots, z_m]^{(n)}) \\
 &= v^{(w(\beta)-m+1)(N-1)} \mathcal{E}_N \frac{1}{\widetilde{\det}_q(I - q\rho'(\gamma))}
 \end{aligned}$$

by quantum MacMahon Master Theorem.

This proves part a) of Theorem 4.1.1. As for part b), first notice that the braid $\overleftarrow{\beta} := \sigma_{i_k}^{\varepsilon_k} \sigma_{i_{k-1}}^{\varepsilon_{k-1}} \dots \sigma_{i_1}^{\varepsilon_1}$ has the closure knot the same as that of β . The Alexander polynomial of K is known to be equal to $\det(I - \overleftarrow{\rho}'(\overleftarrow{\beta}))$, where $\overleftarrow{\rho}$ is the Burau representation, and $\overleftarrow{\rho}'(\overleftarrow{\beta})$ is obtained from $\overleftarrow{\rho}(\overleftarrow{\beta})$ by removing the first row and column. We know that $\mathcal{E}(S_{\pm})$ are the transpose Burau matrices, hence $\overleftarrow{\rho}(\overleftarrow{\beta}) = \mathcal{E}(\rho(\beta))^T$, the transpose of $\mathcal{E}(\rho(\beta))$. The statement now follows.

4.3. The Kashaev invariant

4.3.1. Proof of Theorem 4.1.3.

4.3.1.1. *Completion of $\mathcal{A}_{\varepsilon}$.* Let \mathcal{I} be the left ideal in $\mathcal{A}_{\varepsilon}$ generated by a_1, a_2, \dots, a_k , i.e.

$$\mathcal{I} := a_1 \mathcal{A}_{\varepsilon} + a_2 \mathcal{A}_{\varepsilon} + \dots + a_k \mathcal{A}_{\varepsilon},$$

and let $\hat{\mathcal{A}}_{\varepsilon}$ be the \mathcal{I} -adic completion of $\mathcal{A}_{\varepsilon}$. Using the almost q -commutativity it's easy to see that \mathcal{I} is a two-sided ideal.

LEMMA 4.3.1. *When the closure of $\beta(\gamma)$ is a knot, $\widetilde{\det}_q(I - q\rho'(\gamma))$ belongs to $1 + \mathcal{I}$, and hence $\frac{1}{\widetilde{\det}_q(I - q\rho'(\gamma))}$ belongs to $\hat{\mathcal{A}}_{\varepsilon}$.*

PROOF. It's enough to show that when $a_1 = a_2 = \dots = a_k = 0$, then $\widetilde{\det}_q(I - q\rho'(\gamma)) = 1$, or $\det_q(C) = 0$ for any main minor C of $\rho'(\gamma)$.

Let call *permutation-like* matrix a matrix C where on each row and on each column there is at most one non-zero entry. If, in addition, on each row and on each column there is exactly one non-zero entry, we say that C is non-degenerate. Every non-degenerate permutation-like square matrix C gives rise to a permutation matrix $p(C)$ by replacing all the non-zero entries with 1. It's clear that product of (non-degenerate) permutation-like matrices is a (non-degenerate) permutation-like one. If C is a permutation-like $m \times m$ matrix, and D a main minor, i.e. a submatrix of

type $J \times J$, then D is also permutation like. If, in addition, both C and D are non-degenerate, then $p(C)$ leaves J stable, i.e. $p(C)(J) = J$, since the restriction of $p(C)$ on J is equal to $p(D)$ which leaves J stable.

Also note that if C is degenerate permutation-like right-quantum matrix, then $\det_q(C) = 0$.

When $a_1 = a_2 = \dots = a_k = 0$, each of matrices A_j (whose definition is in subsection 4.2.2.2) is a non-degenerate permutation-like matrix. Hence $C = \rho(\gamma)$ is permutation-like. Note that $p(C)$ is exactly $\bar{\beta}$, the permutation corresponding to β . Because the closure of β is a *knot*, $\bar{\beta} = p(C)$ does not leave any proper subset of $\{1, 2, \dots, m\}$ stable. Hence any main minor D of $\rho'(\gamma)$, which itself is a proper main minor of $C = \rho(\gamma)$, is a degenerate permutation-like matrix. Hence $\det_q(D) = 0$. \square

4.3.1.2. $\hat{\mathcal{A}}_\varepsilon$ and the Habiro ring.

LEMMA 4.3.2. a). *If $f \in \mathcal{A}_\varepsilon$ is divisible by a_j^d for some $1 \leq j \leq k$ and a positive integer d , then $\mathcal{E}(f)$ is divisible by $(1 - zq^r)_q^d$, and hence $\mathcal{E}_N(f)$ is divisible by $(1 - q)_q^d$ for every integer N , not necessarily positive.*

b). *Suppose $n > dk$. Then $\mathcal{E}_N(f)$ is divisible by $(1 - q)_q^d$ for every integer N and every $f \in \mathcal{I}^n$. Hence $\mathcal{E}_N \hat{\mathcal{A}}_\varepsilon \in \widehat{\mathbb{Z}[q]}$.*

PROOF. a). We assume that f is a *monomial* in the variables $a_1, b_1, c_1, a_2, \dots$. Using the almost q -commutativity we move all a_j, b_j, c_j to the right of f , so that $f = g b_j^s c_j^r a_j^d$, for some $g \in \mathcal{A}_\varepsilon$ not containing a_j, b_j, c_j . Note that by Lemma 4.2.4

$$\mathcal{E}(f) = \mathcal{E}(g) \mathcal{E}(b_j^s c_j^r a_j^d)$$

is divisible by $\mathcal{E}(b_j^s c_j^r a_j^d)$. Note that a_j, b_j, c_j are either a_+, b_+, c_+ or a_-, b_-, c_- . Using (4.2.6) and (4.2.7) we see that $\mathcal{E}_N(f)$ is divisible by $(1 - q^l)_q^d$ for some integer l , which, in turn, is always divisible by $(1 - q)_q^d$.

b). Using the fact that generators $a_j, b_j, c_j, 1 \leq j \leq k$ almost q -commute, it's easy to see that \mathcal{I}^n is 2-sided ideal generated by $a_{s_1} a_{s_2} \dots a_{s_n}$, where each s_i is one of $\{1, 2, \dots, k\}$. If $n > dk$, by the pigeon hole principle, there is an index j such $a_{s_1} a_{s_2} \dots a_{s_n}$ is divisible by a_j^d . Now the result follows from part a). \square

From Lemmas 4.3.2 and 4.3.1 we get the following.

COROLLARY 4.3.3. *Suppose N is an integer, not necessarily positive. Then*

$$\mathcal{E}_N \left(\frac{1}{\widetilde{\det}_q(I - q\rho'(\gamma))} \right) \in \widehat{\mathbb{Z}[q]}.$$

4.3.1.3. *Proof of Theorem 4.1.3.* Part a) is a special case of Corollary 4.3.3, with $N = 0$.

For part b) first recall that $\langle K \rangle_N = J'_K(N)|_{q=\exp(2\pi i/N)}$. When $q = \exp(2\pi i/N)$, one has $q^N = 1 = q^0$. Thus $\mathcal{E}_N = \mathcal{E}_0$ when $q = \exp(2\pi i/N)$. One has

$$\begin{aligned} J'_K(N)|_{q=\exp(2\pi i/N)} &= v^{m-1-w(\beta)} \mathcal{E}_N(T)|_{q=\exp(2\pi i/N)} \\ &= v^{m-1-w(\beta)} \mathcal{E}_0(T)|_{q=\exp(2\pi i/N)}, \end{aligned}$$

where

$$T = \frac{1}{\widetilde{\det}_q(I - q\rho'(\gamma))}.$$

4.3.2. The Kashaev invariant for other simple Lie algebra. Fix a simple Lie algebra \mathfrak{g} . For every long knot K , presented by a $1 - 1$ tangle, one can define the \mathfrak{g} -universal invariant $\mathcal{J}_K^{\mathfrak{g}}$, which is a central element in an appropriate completion of quantized universal enveloping algebra $U_v(\mathfrak{g})$, see [Tur94, Law89]. Formally, $\mathcal{J}_{K,\mathfrak{g}}$ is an infinite sum of central elements in $U_v(\mathfrak{g})$:

$$(4.3.1) \quad \mathcal{J}_{K,\mathfrak{g}} = \sum_{n=0}^{\infty} \mathcal{J}_{K,\mathfrak{g}}^{(n)},$$

such that for any finite dimensional simple $U_v(\mathfrak{g})$ -module only the action of a finite number of terms are non-zero. Hence for a finite-dimensional simple module $U_v(\mathfrak{g})$ -module V , $\mathcal{J}_K^{\mathfrak{g}}$ acts as a scalar times the identity. It can be shown that the scalar is a Laurent polynomial in q . Denote this scalar by $J'_{K,\mathfrak{g}}(V)$. One always has

$$J_{K,\mathfrak{g}}(V) = J'_{K,\mathfrak{g}}(V) \dim_q(V),$$

when $J_{K,\mathfrak{g}}(V)$ is the usual quantum invariant of K colored by V , and $\dim_q(V)$ is the quantum dimension, i.e. the invariant of the unknot colored by V .

For any Verma module V_λ of highest weight λ (an element in the weight lattice), the action of each of $\mathcal{J}_{K,\mathfrak{g}}^{(n)}$ is still in $\mathcal{R} = \mathbb{Z}[q^{\pm 1}]$, but in general infinitely many of them are non-zero. In this case $J'_{K,\mathfrak{g}}(V_\lambda)$ is an infinite series (sum). In a future work we will show that $J'_{K,\mathfrak{g}}(V_\lambda) \in \widehat{\mathbb{Z}[q]}$; the special case when $\mathfrak{g} = \mathfrak{sl}_2$ has been proved here by Corollary 4.3.3.

Note that if the weight λ is dominant, then

$$J'_{K,\mathfrak{g}}(V_\lambda) = J'_{K,\mathfrak{g}}(W_\lambda),$$

where W_λ is the finite dimensional $U_v(\mathfrak{g})$ module with highest weight λ . The reason is both are the scalar of the same scalar operator acting on V_λ and its quotient W_λ . In this case $J'_{K,\mathfrak{g}}(V_\lambda)$ is a Laurent polynomial in q . It is known that $\mathcal{R} = \mathbb{Z}[q^{\pm 1}] \subset \widehat{\mathbb{Z}[q]}$, see [Hab02].

Due to the Weyl symmetry, we see that if w is in the Weyl group, then $J'_{K,\mathfrak{g}}(V_\lambda) = J'_{K,\mathfrak{g}}(V_{w \cdot \lambda})$, where $w \cdot \lambda$ is the dot action of the Weyl group, see [Hum78]. If λ is not fixed (under the dot action) by any element of the Weyl group, then $\lambda = w \cdot \mu$ for some dominant μ , and hence $J'_{K,\mathfrak{g}}(V_\lambda) = J'_{K,\mathfrak{g}}(V_\mu)$. In this case $J'_{K,\mathfrak{g}}(V_\lambda)$ might be still an infinite series, but it is equal to a Laurent polynomial, which is $J'_{K,\mathfrak{g}}(V_\mu)$ in the Habiro ring $\widehat{\mathbb{Z}[q]}$.

The more interesting, and less understood case is when λ is fixed by an element of the Weyl group, i.e. λ is on a wall of a *shifted* Weyl chamber. Among them there is one special weight, namely $\lambda = -\delta$, where δ is the half-sum of positive roots, since $-\delta$ is the only element invariant by *all* elements of the Weyl group. When $\mathfrak{g} = \mathfrak{sl}_2$, $V_{-\delta}$ is V_0 in section 4.2, and $J'_{K,\mathfrak{g}}(V_{-\delta})$ is the Kashaev invariant in this case, according to Theorem 4.1.3. Note that $V_{-\delta}$ is always infinite-dimensional and irreducible; it's certainly a very special $U_v(\mathfrak{g})$ -module.

Thus a natural generalization of the Kashaev invariant to other simple Lie algebra is $J'_{K,\mathfrak{g}}(V_{-\delta})$. More precisely, let's define the \mathfrak{g} -Kashaev invariant by

$$\langle K \rangle_N^{\mathfrak{g}} := J'_{K,\mathfrak{g}}(V_{-\delta})|_{q=\exp(2\pi i/N)}.$$

And we suggest the following \mathfrak{g} -volume conjecture

$$\lim_{N \rightarrow \infty} \frac{|\langle K \rangle_N^{\mathfrak{g}}|}{N} = c_{\mathfrak{g}} \text{Vol}(K),$$

where $c_{\mathfrak{g}}$ is a constant depending only on the simple Lie algebra \mathfrak{g} .

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