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Applications of Rearrangement functions in the equation $\Delta u(x) = K(x)e^{u(x)}$

1 Introduction

Recently in the paper [6], M.R.Posteraro has investigated the equation

$$\Delta u(x) = e^{u(x)}$$

on a bounded domain Ω in $\mathbb{R}^2$, with boundary condition

$$\lim_{x \to \partial\Omega} u(x) = +\infty.$$

Assuming the equation has a solution in $C^2(\Omega)$, Posteraro used rearrangement functions and isoperimetric inequalities to obtain the following estimate

$$\min_{x\in\Omega} u(x) \geq \log(8\pi/|\Omega|).$$

In this paper we consider the equation

$$\Delta u(x) = K(x)e^{u(x)}$$

where $u \in W_0^{1,2}(\Omega)$; $0 < K(x) \leq a$ and $K \in C^\infty(\bar{\Omega})$.

In section 2 we investigate the continuity of the solution; in section 3 we apply rearrangement functions and isoperimetric inequalities for the above equation; in section 4, using some techniques of differential inequalities we obtain an estimate of the rearrangement of u, after that we show that using that estimate we are able to extend the result of POSTERARO; in the Appendix we state several theorems which we used, without proof. Some places which need detail computations are numbered and explained in the Footnotes.

2 Continuity of the solution

Let Ω be a bounded domain in $I\!R^2$ with $\partial\Omega \in C^2$ and $K \in C^\infty(\bar{\Omega})$ with $K(\Omega) \subset (0, a]$, we consider the solution $u \in W_0^{1,2}(\Omega)$ of the equation

$$\Delta u(x) = K(x)e^{u(x)}. \tag{2.1}$$

We use the notation $\partial\Omega \in C^2$ to mean Ω has boundary belonging to the class C^2, and similarly for other boundary classes; c, c_1, c_2...are constants.

The existence of solutions of the equation (2.1) is known, see ([7], page 434).

The following section is based on a remark in ([7], page 437).

LEMMA 2.1 *Consider the equation $\Delta u(x) = f(x)$. Suppose that $u \in W^{1,2}(\Omega)$, ($\partial\Omega \in C^2$) is a solution of the equation. If $f \in W^{k,2}(\Omega)$, $k \geq 0$ then $u \in W^{k+2,2}(\Omega)$.*

Proof. We prove by induction.
When $k = 0$, we have $u \in W^{2,2}(\Omega)$ according to ([2], theorem 8.12; also see [3], theorem 10.1 page 188).
Suppose the conclusion of the lemma is true for $k \geq 0$. Suppose $f \in W^{k+1,2}(\Omega)$. Since $u \in W^{1,2}(\Omega)$ is a solution, we have

$$-\int_\Omega D_i u D_i v dx = \int_\Omega f v dx \ , \forall v \in C_0^\infty(\Omega).$$

Replace v by $D_l v$ we have

$$-\int_\Omega D_i u D_{il} v dx = \int_\Omega f D_l v dx.$$

Taking integration by parts, we get

$$-\int_\Omega D_{il} u D_i v dx = \int_\Omega D_l f v dx \ , \forall v \in C_0^\infty(\Omega).$$

This implies $D_l u \in W^{1,2}(\Omega)$ is a solution of the equation $\Delta u = D_l f$, with $D_l f \in W^{k,2}(\Omega)$. By the induction assumption we have $D_l u \in W^{k+2,2}(\Omega)$, which implies $u \in W^{k+3,2}(\Omega)$, we have the desired result.$\square$

PROPOSITION 2.1 *Suppose (2.1) is satisfied and* $u \in W^{k,2}(\Omega)$, $k \geq 2$. *Then* $Ke^u \in W^{k-1,2}(\Omega)$.

Proof. We have

$$D_i(Ke^u) = D_i K e^u + K e^u D_i u^{(1)}.$$

According to Sobolev Imbedding Theorem $W^{k,2}(\Omega) \hookrightarrow C^{k-2}(\bar{\Omega})$ (see the Appendix). Because $D_i u \in W^{k-1,2}(\Omega)$ we have $Ke^u D_i u \in W^{k-2,2}(\Omega)^{(2)}$. At the same time $D_i K e^u \in C^{k-2}(\bar{\Omega})$. This implies $D_i(Ke^u) \in W^{k-2,2}(\Omega)$, hence $Ke^u \in W^{k-1,2}(\Omega)$.$\square$

LEMMA 2.2 *Suppose (2.1) is satisfied. Then we have* $u \in W^{k,2}(\Omega)$, $\forall k \geq 2$. *Hence* $u \in C^\infty(\bar{\Omega})$.

Proof. We prove by induction.

When $k = 2$, according to ([7], theorem 1.3) we have $e^u \in L^2(\Omega)$. Whence Ke^u is a function belongs to $L^2(\Omega)$. As in the proof of lemma 2.1, we have $u \in W^{2,2}(\Omega)$.

Suppose that the conclusion of the lemma is true for k and we have $u \in$

$W^{k,2}(\Omega)$. According to proposition 2.1 we have $Ke^u \in W^{k-1,2}(\Omega)$. Following proposition 2.1 we have $u \in W^{k+1,2}(\Omega)$. Thus $u \in W^{k,2}(\Omega)$, $\forall k \geq 2$. This implies, according to Sobolev Imbedding Theorem, $u \in C^\infty(\bar{\Omega})$ (see the Appendix). We have the desired result.□

3 Applications of rearrangement functions and ioperimetric inequalities

This section is based on Talenti[5] and Mossino([4], chapter 1).

Let u be a measurable function on Ω. The distribution function of u is defined to be

$$\mu : I\!R \to [0, |\Omega|]$$

$$\mu(t) = |\{x \in \Omega, u(x) < t\}|$$

where the notation $|.|$ denote the 2-dimensional Lebesgue measure.

The increasing (1-dimensional) rearangement function of u is defined to be

$$u_* : [0, |\Omega|] \to I\!R \cup \{-\infty\} \cup \{+\infty\}$$

$$\begin{cases} u_*(s) & = & \inf\{t \in I\!R, \mu(t) > s\} \text{ if } s \in [0, |\Omega|) \\ u_*(|\Omega|) & = & ess \sup_\Omega u \end{cases}.$$

For convenience, we denote $G_t = \{x \in \Omega : u(x) = t\}$ and $H_t = \{x \in \Omega : u(x) < t\}$.

Proposition 3.1 [(3)] *One has that :*

(i). μ is an increasing function.

(ii). $u_*(s) \neq \pm\infty$ if $s \in (0, |\Omega|)$.

(iii). u_* is an increasing function.

(iv). $u_*(0) = ess \inf_\Omega u$.

Lemma 3.1 *Let* Ω_0 *be a bounded domain such that* $\partial\Omega_0 \in C^1$ *and* $u \in C^1(\bar{\Omega}_0)$*. We can extend* u *to become a Lipschitz function on* $I\!R^2$ *which still preserves its value on* Ω_0*.*

Proof. Let B be a ball such that $\Omega_0 \subset\subset B$. According to (V.P.Mikhailov, *Partial Differential Equations*, Mir, Moscow, 1978 , page 129, theorem 1), there exists an extension u_1 of u such that $u_1 = u$ on Ω_0 and $u_1 \in C^1(\bar{B})$ with compact support.

Let

$$u_2(x) = \begin{cases} u_1(x) & x \in \bar{B} \\ 0 & x \in I\!R^2 \setminus \bar{B} \end{cases}$$

We will prove that $u_2 \in C^{0,1}(I\!R^2)$. Since $u_1 \in C^1(\bar{B})$ and B is a convex set, we have $u_1 \in C^{0,1}(\bar{B})$ with Lipschitz constant c_1. Consider $x,\, y \in I\!R^2$.

If $x,\, y \in \bar{B}$ we have

$$|u_2(x) - u_2(y)| = |u_1(x) - u_1(y)| \leq c_1|x - y|.$$

If $x \in I\!R^2 \setminus \bar{B}$ and $y \in \bar{B}$ we have

$$|u_2(x) - u_2(y)| = |u_2(y)| \leq c_2 \,\forall y \in \bar{B}.$$

We have two cases: if $y \in supp(u_1)$ then $\forall x \in I\!R^2 \setminus \bar{B}$ we have $|x - y| \geq d(supp(u_1), \partial B) = c_3 > 0$, hence $\forall y \in supp(u_1)$ we have $|u_1(y)| \leq c_4|x - y|$; if $y \in \bar{B} \setminus supp(u_1)$ then $u_2(y) = u_1(y) = 0$.

If $x, y \in I\!R^2 \setminus \bar{B}$ then $u_2(x) = u_2(y) = 0$.

Thus u_2 is Lipschitz on $I\!R^2$.$\square$

Consider a solution u of (2.1). As said above, $u \in W_0^{1,2}(\Omega) \cap C^\infty(\bar{\Omega})$. We use Sard theorem (see the Appendix). The point $x \in \Omega$ such that $|\nabla u(x)| = 0$ is called a critical point of u, the remaining points are called regular points. The value $t \in I\!R$ such that $u^{-1}(\{t\})$ contains at least a critical point of

u is called a critical value, the remaining values are called regular values. According to Sard theorem, since $u \in C^\infty(\Omega)$, the set of critical values of u has measure 0. Besides if t is a regular value of u then we also have $u^{-1}(\{t\})$ is of zero measure.

Consider a regular value t_0 of u. Let $K_t = \{x \in \Omega : t_0 \leq u(x) \leq t\}$. With sufficiently small $h > 0$,K_{t_0+h} doesn't contain any critical points of u. Indeed , suppose that $\{h_n\} \searrow 0$ and $K_{t_0+h_n}$ contains a critical point x_n. Since the set K_{t_0+h} is compact there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ convergent to x_0. We have

$$t_0 \leq u(x_{n_k}) \leq t_0 + h_{n_k}\ \forall k.$$

This implies $\lim_{k\to+\infty} u(x_{n_k}) = u(x_0) = t_0$.
On the other hand $\lim_{k\to+\infty} |\nabla u(x_{n_k})| = |\nabla u(x_0)|$
$= 0$. So x_0 is a critical point corresponding to t_0, this is a contradiction.

Quite similarly, we can prove for $h < 0$. So if t_0 is a regular value then there are only regular values in a neighborhood of t_0. In other words, the set of regular values of u is open. As known, this set must be a union of at most countable number of open intervals in $I\!R$. Call (t_1, t_2) such an interval which contains t_0, and let $t \in (t_1, t_2)$.

First, suppose $t \geq t_0$. Call u_2 a Lipschitz extension of u from K_t to $I\!R^2$ as in lemma 3.1 (more accurately, let $\Omega_0 = \{x \in \Omega, d(x, \partial\Omega) > \frac{1}{2}d(K_t, \partial\Omega)\}$ then $K_t \subset \Omega_0$ and $\partial\Omega_0 \in C^2$ and we extend from Ω_0).
Let $g(x) = \chi_{K_t}(x)/|\nabla u_2(x)| = \chi_{K_t}(x)/|\nabla u(x)|$. Applying Federer theorem (see the Appendix) for g and u_2 we get

$$\int_{I\!R^2} \frac{\chi_{K_t}(x)}{|\nabla u_2(x)|}|\nabla u_2(x)|dx = \int_{I\!R}\int_{\{x\in\Omega: u_2=\theta\}} \frac{\chi_{K_t}(x)}{|\nabla u_2(x)|}d\Gamma d\theta$$

where Γ is 1-dimensional surface measure. This implies

$$|K_t| = \int_{I\!R^2} \chi_{K_t}(x)dx \quad = \quad \int_{I\!R}\int_{\{x\in\Omega: u_2=\theta\}} \frac{\chi_{K_t}(x)}{|\nabla u_2(x)|}d\Gamma d\theta$$

$$= \int_{t_0}^{t}\int_{G_\theta}\frac{1}{|\nabla u(x)|}d\Gamma d\theta$$

Since t is a regular value, we have $|G_t| = 0$. This implies

$$\begin{aligned}\mu(t)-\mu(t_0) = |H_t| - |H_{t_0}| &= |\{x\in\Omega : t_0 \le u(x) < t\}|\\ &= |K_t|\\ &= \int_{t_0}^{t}\int_{G_\theta}\frac{1}{|\nabla u(x)|}d\Gamma d\theta.\end{aligned}$$

Since μ is increasing function, it has derivative almost everywhere, hence we can take derivatives of both sides of the above equality, which imply

$$\mu^{'}(t) = \int_{G_t}\frac{1}{|\nabla u(x)|}d\Gamma > 0 \text{ a.e. } t\in[t_0,t_2). \tag{3.1}$$

Similarly, we can prove (3.1) for $t\in(t_1,t_0]$. As such this equality is true for almost all $t\in(t_1,t_2)$, and morever is true for lamost all $t\in I\!R$.

Consider a regular value t of u, according to ([1] page 38) we have

$$\int_{H_t}\Delta u dx = \int_{H_t} div(\nabla u)dx = \int_{G_t}\nabla u.n d\Gamma$$

here $n = \nabla u/|\nabla u|$ (oriented toward $\Omega\setminus H_t$). This implies

$$\int_{H_t}\Delta u dx = \int_{G_t}\nabla u.\frac{\nabla u}{|\nabla u|}d\Gamma = \int_{G_t}|\nabla u|d\Gamma. \tag{3.2}$$

Applying Schwartz inequality, we get

$$\int_{G_t}\sqrt{|\nabla u|}\frac{1}{\sqrt{|\nabla u|}}d\Gamma \le [\int_{G_t}|\nabla u|d\Gamma\int_{G_t}\frac{1}{|\nabla u|}d\Gamma]^{1/2}$$

or, because of (3.1)

$$\Gamma(G_t)\le[\mu^{'}(t)\int_{G_t}|\nabla u|d\Gamma]^{1/2}$$

(Γ is 1-dimensional surface measure, also is 1-dimensional Lebesgue measure, see ([1], page 37)). The classical isoperimetric inequality ([4] chapter 1.4, lemme 4.2) gives us

$$\Gamma(G_t)\ge 2\pi^{1/2}\mu^{1/2}(t).$$

Hence we have

$$4\pi\mu(t) \leq \mu^{'}(t) \int_{G_t} |\nabla u| d\Gamma.$$

Together with (3.2) and (2.1) we obtain

$$4\pi\mu(t) \leq \mu^{'}(t) \int_{H_t} \Delta u dx = \mu^{'}(t) \int_{H_t} Ke^u dx \text{ h.h. } t \in I\!R. \tag{3.3}$$

Lemma 3.2 *One has*

$$\int_{H_t} Ke^u dx = \int_0^{|H_t|} (Ke^u)_* d\tau.$$

Proof. We have the following general result: if $E \subset \Omega$ and v is measurable on Ω then

$$\int_E v(x)dx \geq \int_0^{|E|} v_*(s)ds.$$

And the equality occur if and only if $(v|_E)_* = u_*|_{(0,|E|)}$ (see [4], chapter 1.3, lemme 3.1). So we only need to show that the inequality occur in this case. Let $s \in (0, |H_t|)$, we have

$$\begin{aligned}
(Ke^u|_{H_t})_*(s) &= \inf\{\theta : |\{x \in \Omega, Ke^{u(x)} < \theta \wedge u(x) < t\}| > s\} \\
&= \inf\{\theta : |\{x \in \Omega, u(x) < \log(\theta/K) \wedge u(x) < t\}| > s\}
\end{aligned}$$

v $s < |H_t|$, we have

$$\begin{aligned}
& |\{x \in \Omega, u(x) < \log(\theta/K) \wedge u < t\}| > s \\
\iff & |\{x \in \Omega, u(x) < \log(\theta/K)\}| > s \\
\iff & |\{x \in \Omega, Ke^{u(x)} < \theta\}| > s.
\end{aligned}$$

Thus $(Ke^u|_{H_t})_* = (Ke^u)_*|_{(0,|H_t|)}$.□

Applying the above lemma to (3.3) we get

$$4\pi\mu(t) \leq \mu^{'}(t) \int_0^{|H_t|} (Ke^u)_*(\tau)d\tau = \mu^{'}(t) \int_0^{\mu(t)} (Ke^u)_*(\tau)d\tau \text{ h.h. } t \in I\!R. \tag{3.4}$$

LEMMA 3.3 (4) *Let u, v be measurable functions on Ω. One has*

(i). If $u \le v$ a.e. then $u_* \le v_*$ everywhere on $(0, |\Omega|)$.

(ii). If $c > 0$ then $(cu)_* = cu_*$ on $(0, |\Omega|)$.

(iii). $(e^u)_* = e^{u_*}$ on $(0, |\Omega|)$.

About this lemma see ([4] chapter 1.1, theoreme 1.2).

Applying lemma 3.3 we get $(Ke_u)_* \le ae^{u_*}$, hence from (3.4) we have

$$4\pi\mu(t) \le a\mu^{'}(t)\int_0^{\mu(t)} e^{u_*(\tau)}d\tau \text{ a.e. } t \in I\!R. \tag{3.5}$$

Since the set Ω is connected and $u \in C(\Omega)$, $u(\Omega)$ is an open interval with two endpoints $\inf_\Omega u$ and $\sup_\Omega u$.

LEMMA 3.4 *One has*

$$u_*^{'}(\mu(t)) = \frac{1}{\mu^{'}(t)} \text{ a.e. } t \in [\inf_\Omega u, \sup_\Omega u).$$

Proof. Since u_*, μ are increasing functions, they are differentiable almost everywhere , hence $u_* \circ \mu$ is also differentiable almost everywhere. At the points where $u_* \circ \mu$ is differentiable, we have

$$\begin{aligned}
(u_* \circ \mu)^{'}(t) &= \lim_{h \to 0^+} \frac{u_*(\mu(t+h)) - u_*(\mu(t))}{h} \\
&= \lim_{h \to 0^+} \frac{\inf\{\theta, \mu(\theta) > \mu(t+h)\} - \inf\{\theta, \mu(\theta) > \mu(t)\}}{h}
\end{aligned}$$

We notice that if u is continuous then μ is striclty increasing on $[\inf_\Omega u, \sup_\Omega u)$. Indeed, suppose $\theta > t, t \in [\inf_\Omega u, \sup_\Omega u)$, we have

$$\begin{aligned}
\mu(\theta) - \mu(t) &= |H_\theta| - |H_t| \\
&= |\{x \in \Omega : t \le u(x) < \theta\}| \\
&\ge |\{x \in \Omega : t < u(x) < \theta\}|
\end{aligned}$$

$\{x \in \Omega : t < u(x) < \theta\}$ is an open nonempty set, hence must have positive measure , this implies μ is strictly increasing. Similarly, if $h > 0$ sufficiently small then $\mu(\theta) > \mu(t+h) \Leftrightarrow \theta > t+h$.
From this we have $\inf\{\theta : \mu(\theta) > \mu(t)\} = \inf\{\theta : \theta > t\} = t$ and $\inf\{\theta : \mu(\theta) > \mu(t+h)\} = \inf\{\theta : \theta > t\} = t+h$. Thus

$$(u_* \circ \mu)^{'}(t) = \lim_{h \to 0} \frac{t+h-t}{h} = 1.$$

From this we get the desired result.□

Applying lemma 3.4 to (3.5) we get for almost all $t \in$ $[\inf_\Omega u, \sup_\Omega u)$

$$\frac{1}{\mu^{'}(t)} \leq \frac{a}{4\pi\mu(t)} \int_0^{\mu(t)} e^{u_*(\tau)} d\tau^{(5)}$$

or

$$u_*^{'}(\mu(t)) \leq \frac{a}{4\pi\mu(t)} \int_0^{\mu(t)} e^{u_*(\tau)} d\tau.$$

Let $s = \mu(t)$, for almost all $s \in [0, |\Omega|]$ we have ([6], page 447)

$$u_*^{'}(s) \leq \frac{a}{4\pi s} \int_0^s e^{u_*(\tau)} d\tau \text{ h.h. } s \in [0, |\Omega|]. \tag{3.6}$$

LEMMA 3.5 *$u_*(s)$ is right-continuous on $[0, |\Omega|)$.*

Proof. Since u is bounded on Ω, $u_*(s)$ is finite. Given $s_0 \in [0, |\Omega|)$ and $h > 0$ we have

$$\begin{aligned} 0 \leq u_*(s_0+h) - u_*(s_0) &= \inf\{t : |H_t| > s_0 + h\} - u_*(s_0) \\ &= \inf\{t - u_*(s_0) : |H_t| > s_0 + h\} \\ &= \inf\{t : |H_{t+u_*(s_0)}| > s_0 + h\}. \end{aligned}$$

We need to prove that

$$\forall \epsilon > 0, \exists \delta > 0, 0 < h < \delta \Rightarrow \inf\{t : |H_{t+u_*(s_0)}| > s_0 + h\} < \epsilon.$$

On the other hand

$$\inf\{t : |H_{t+u_*(s_0)}| > s_0 + h\} < \epsilon \Longleftrightarrow \exists t_0, 0 \le t_0 < \epsilon : |H_{t_0+u_*(s_0)}| > h + s_0$$

we have $u_*(s_0) = \inf\{\theta, |H_\theta| > s_0\}$ this implies

$$\forall \epsilon > 0, \exists \theta_\epsilon \text{ such that } 0 \le \theta_\epsilon - u_*(s_0) < \epsilon \text{ and } |H_{\theta_\epsilon}| > s_0.$$

Let $t_\epsilon = \theta_\epsilon - u_*(s_0)$ then

$$\forall \epsilon > 0, \exists t_\epsilon, 0 \le t_\epsilon < \epsilon \text{ v } |H_{t_\epsilon+u_*(s_0)}| > s_0.$$

Choose $0 < \delta < |H_{t_\epsilon+u_*(s_0)}| - s_0$ then

$$\forall \epsilon > 0, \exists \delta > 0, 0 < h < \delta \Rightarrow \exists t_\epsilon, 0 \le t_\epsilon < \epsilon, |H_{t_\epsilon+u_*(s_0)}| > h + s_0.$$

We get the desired result.□

Consider the equation (2.1). Since $K(x)e^{u(x)} \ge 0$, according to the maximum principle ([2], chapter 3.1) we have $u \le 0$ on Ω. Then apply the Polya-Szego theorem we get $u_* \in W^{1,2}(c, |\Omega|)\, \forall c > 0$ (see [4], chapter 1.4, theoreme 4.1). According to Sobolev Imbedding Theorem $W^{1,2}(c, |\Omega|) \hookrightarrow C^{0,1/2}[c, |\Omega|]$ (see the Appendix), thus $u_* \in C^{0,1/2}(0, |\Omega|]$. Together with lemma 3.5 we imply that $u_* \in C^{0,1/2}[0, |\Omega|]$.

Now let

$$U(s) = \int_0^s e^{u_*(\tau)} d\tau \tag{3.7}$$
$$M = \min_{x\in\Omega} u(x)$$

we have $U \in C^1[0, |\Omega|]$, besides

$$\begin{aligned} U(0) &= 0 \\ U'(s) &= e^{u_*(s)} \\ U''(s) &= u_*'(s) e^{u_*(s)} \text{ a.e. } s \in [0, |\Omega|]. \end{aligned}$$

From (3.6) we imply

$$U^{''}(s) \leq \frac{a}{4\pi s} e^{u_*(s)} \int_0^s e^{u_*(\tau)} d\tau \leq \frac{a}{4\pi s} U^{'}(s) U(s) \text{ a.e. } s \in (0, |\Omega|]. \qquad (3.8)$$

Consider (3.8)

$$\begin{aligned} U^{''}(s) &\leq \frac{a}{4\pi s}\frac{1}{2}(U^2)^{'}(s) \\ \Rightarrow \quad \frac{8\pi}{a} s U^{''}(s) &\leq (U^2)^{'}(s) \\ \Rightarrow \quad \frac{8\pi}{a}\int_0^s \tau U^{''}(\tau) d\tau &\leq \int_0^s (U^2)^{'}(\tau) d\tau \\ &= U^2(s) - U^2(0) = U^2(s) \; \forall s \in [0, |\Omega|]. \qquad (3.9) \end{aligned}$$

The question is that whether we can take integration by parts. Let $F(s) = sU^{'}(s) = se^{u_*(s)}$ then F is an absolute continuous function on $[0, |\Omega|]$. Indeed, suppose $(a_i, b_i), 1 \leq i \leq n$ is a family of disjoint subintervals of $[0, |\Omega|]$. Since $f(s) = s \in C^{0,1}[0, |\Omega|]$ and $u_* \in C^{0,1/2}[0, |\Omega|]$ we have $se^{u_*} \in C^{0,1/2}[0, |\Omega|]^{(6)}$, hence

$$\begin{aligned} \sum_{i=1}^n |F(b_i) - F(a_i)| &= \sum_{i=1}^n |b_i e^{u_*(b_i)} - a_i e^{u_*(a_i)}| \\ &\leq \sum_{i=1}^n c_1 (b_i - a_i)^{1/2} \\ &\leq c_1 \sum_{i=1}^n (b_i - a_i)^{1/2} \\ &\leq c_1 \sqrt{n} [\sum_{i=1}^n (b_i - a_i)]^{1/2}. \end{aligned}$$

According to the definition in (Hoang Tuy, *Giai Tich Hien Dai*, vol 1, Ha Noi, 1977, page 168) we have F is an absolute continuous function and

$$\begin{aligned} F(s) &= F(0) + \int_0^s F^{'}(\tau) d\tau \\ &= 0 + \int_0^s \tau U^{''}(\tau) + U^{'}(\tau) d\tau \end{aligned}$$

$$
\begin{aligned}
&= \int_0^s \tau U^{''}(\tau)d\tau + \int_0^s U^{'}(\tau)d\tau \\
&= \int_0^s \tau U^{''}(\tau)d\tau + U(s) - U(0) \\
&= \int_0^s \tau U^{''}(\tau)d\tau + U(s)
\end{aligned}
$$

this implies

$$sU^{'}(s) - U(s) = \int_0^s \tau U^{''}(\tau)d\tau.$$

Substitute into (3.9) we get

$$\frac{8\pi}{a}[sU^{'}(s) - U(s)] \leq U^2(s),\ \forall s \in [0, |\Omega|].$$

This implies

$$\begin{cases} U^{'}(s) \leq \dfrac{U^2(s) + \frac{8\pi}{a}U(s)}{\frac{8\pi}{a}s},\ \forall s \in (0, |\Omega|] \\ U(0) = 0 \\ U^{'}(0) = e^M \\ U \in C^1[0, |\Omega|] \end{cases} . \tag{3.10}$$

4 Estimating the solutions

This section is based on some techniques of differential inequalities in ([8]).

Consider the following system

$$\begin{cases} u^{'}(t) \leq \dfrac{u^2(t) + \alpha u(t)}{\alpha t},\ t \in (0, \beta); \alpha, \beta > 0 \\ u(0) = 0 \\ u^{'}(0) = \gamma,\ \gamma > 0 \\ u \in C^1[0, \beta] \end{cases} . \tag{4.1}$$

The results in section 3 (system (3.10)) show that U will satisfy (4.1) with $\alpha = 8\pi/a,\ \beta = |\Omega|,\ \gamma = e^M$.

Consider the following system

$$\begin{cases} u^{'}(t) = \dfrac{u^2(t) + \alpha u(t)}{\alpha t}, \ t \in (0, \beta) \\ u(0) = 0 \\ u \in C^1[0, \beta) \end{cases} . \tag{4.2}$$

By direct computations we find a solution

$$r(t) = \frac{\alpha t}{\beta - t}$$

of (4.2).

Lemma 4.1 *r is a maximal solution of (4.2), meaning if v is a solution of (4.2) then we have*

$$v(t) \leq r(t), \ \forall t \in [0, \beta).$$

Proof. First we prove : if $\exists t_0 \in [0, \beta)$ such that $r(t_0) > v(t_0)$ then $r(t) \geq v(t) \, \forall t \in [0, t_0]$.
Let $w(t) = r(t) - v(t)$.
We have

$$\begin{cases} w(0) = 0 \\ w(t_0) > 0 \\ w \in C^1[0, t_0]. \end{cases}$$

This implies that there is $t_1 \in [0, t_0]$ such that $w(t_1) = \min_{[0,t_0]} w(t)$. We prove $w(t_1) \geq 0$. Suppose the contrary, $w(t_1) < 0$, this implies $0 < t_1 < t_0$, hence we must have $w'(t_1) = 0$.
On the other hand

$$\begin{cases} r^{'}(t) &= \dfrac{r^2(t) + \alpha r(t)}{\alpha t} \\ v^{'}(t) &= \dfrac{v^2(t) + \alpha v(t)}{\alpha t} \end{cases}$$

This implies

$$(r^{'} - v^{'})(t) = \frac{(r(t) - v(t))(r(t) + v(t) + \alpha)}{\alpha t}$$

or

$$w^{'}(t) = \frac{w(t)(r(t) + v(t) + \alpha)}{\alpha t}.$$

Since $w^{'}(t_1) = 0$ and $w(t_1) < 0$, we must have $r(t_1) + v(t_1) + \alpha = 0$.
But $v(t_1) > r(t_1) = \dfrac{\alpha t_1}{\beta - t_1} > 0$, we have a contradiction.

Now consider an arbitrary point t_0 on $[0, \beta)$. Let $r_{\beta'}(t) = \dfrac{\alpha t}{\beta' - t}$, $t_0 < \beta^{'} < \beta$. Since $v(t)$ is finite at $\beta^{'}$ while $\lim_{t \to \beta'-} r_{\beta'}(t) = +\infty$, we must have

$$\exists t_1, t_0 < t_1 < \beta^{'} : r_{\beta'}(t_1) > v(t_1).$$

From the above results we imply $r_{\beta'}(t) > v(t)$, $\forall t \in [0, t_1]$ this implies $r_{\beta'}(t_0) > v(t_0)$. This implies

$$v(t_0) \leq \inf_{t_0 < \beta' < \beta} \frac{\alpha t_0}{\beta' - t_0} = \frac{\alpha t_0}{\beta - t_0} = r(t_0).$$

Thus r is the maximal solution of (4.2).□

Lemma 4.2 *Suppose that u is a solution of* (4.1)*. If $\beta = \alpha/\gamma$ then*

$$u(t) \leq r(t),\ \forall t \in [0, \beta).$$

Proof. First consider $\beta^{'} < \beta$. We observe that $\exists \{\epsilon_n\}$, $\epsilon_n \searrow 0$ such that $u(\epsilon_n) \leq \dfrac{\alpha \epsilon_n}{\beta' - \epsilon_n}$. Indeed, suppose the contrary , $\exists \delta > 0$ such that $\forall \epsilon < \delta$, $u(\epsilon) > \dfrac{\alpha \epsilon}{\beta' - \epsilon}$, we have

$$u(\epsilon) > \frac{\alpha \epsilon}{\beta' - \epsilon}$$

or

$$\frac{u(\epsilon)}{\epsilon} > \frac{\alpha}{\beta' - \epsilon}.$$

Let $\epsilon \to 0$, since $u(0) = 0$ and $u'(0) = \gamma$ we get $\gamma \geq \alpha/\beta'$, this contradicts the above assumption.

With fixed n, we consider the differential equation

$$\begin{cases} u'(t) = \dfrac{u^2(t) + \alpha u(t)}{\alpha t}, \ t \in (\epsilon_{n+1}, \beta') \\ u(\epsilon_n) = \dfrac{\alpha\epsilon_n}{\beta' - \epsilon_n} \\ u \in C^1(\epsilon_{n+1}, \beta') \end{cases}. \tag{4.3}$$

We imply immediately that $\dfrac{\alpha t}{\beta' - t}$ is the maximal solution of (4.3) (notice that lemma 4.1 doesn't change if we replace in (4.2) the initial point to be ϵ_n and the initial value to be $\dfrac{\alpha\epsilon_n}{\beta' - \epsilon_n}$).

Applying the comparison theorem 1.4.1 in [8] we imply that every solution u of (4.1) satisfies

$$u(t) \leq \frac{\alpha t}{\beta' - t}, \ \forall t \in [\epsilon_n, \beta').$$

Let $n \to \infty$ we get $u(t) \leq \dfrac{\alpha t}{\beta' - t}$, $\forall t \in [0, \beta')$. Fix $t \in [0, \beta)$, we get

$$u(t) \leq \frac{\alpha t}{\beta' - t}, \ \forall \beta' \in (t, \beta)$$

this implies

$$u(t) \leq \inf_{t<\beta'<\beta} \frac{\alpha t}{\beta' - t} = \frac{\alpha t}{\beta - t} = r(t).$$

Thus we get the desired result.□

From lemma 4.2 we deduce the following result

$$\int_0^s e^{u_*(\tau)} d\tau \leq \frac{\frac{8\pi}{a} s}{\frac{8\pi}{ae^M} - s}, \ \forall s \in [0, \frac{8\pi}{ae^M}). \tag{4.4}$$

This inequality gives us an estimate of the rearrangement function of u.

In what follows we show that in the case of the problem of POSTERARO we can go farther and are able to extend the result of this author.

Suppose $u \in C^2(\Omega)$ is a solution of the equation

$$\Delta u(x) = K(x)e^{u(x)} \tag{4.5}$$

satisfying $\lim_{x\to\partial\Omega} u(x) = +\infty$, where $K \in C^\infty(\Omega)$ and $K(x) \leq a, a > 0$.

In the case $K = 1$, according to [6], the existence of solutions of (4.5) is known. Here we assume that (4.5) has a solution.

PROPOSITION 4.1 *If u is a solution of* (4.5) *then* $u \in W^{k,2}(\Omega')$; $\forall k \geq 2$, $\forall \Omega' \subset\subset \Omega$, *hence* $u \in C^\infty(\Omega)$.

Proof. The proof of this proposition is similar to that of section 2, only replacing Ω by Ω'.(7) $\square$

We notice that all the results from section 3 to the estimate (4.4) are still true for the equation (4.5) without changes in the arguments, except for the paragraphs which we will make clear in what follows.

First is the paragraph which proves $u_* \in C^{0,1/2}[0, |\Omega|]$ after lemma 3.5.

PROPOSITION 4.2 $u_* \in C^{0,1/2}[0, |\Omega|)$.

Proof. We have $\Omega = \cup_{t\in\mathbb{R}} H_t$. With fixed t, there is $t' > t$, such that $H_t \subset\subset H_{t'}$. Indeed since $\lim_{x\to\partial\Omega} u(x) = +\infty$ we have $\exists t'$ such that $H_{t'} \neq \bar{H}_t$.

Let $v(x) = u(x) - t'$. Whence

$$\begin{cases} v(x) < t - t' \text{ on } H_t \\ t - t' \leq v(x) < 0 \text{ on } H_{t'} \setminus H_t \\ v \in C^\infty(\Omega) \end{cases} .$$

Since $\partial H_t = G_t$ is an infinitely differentiable manifold ([1], page 35, more accurately, this is true for almost all t), there exists a function $\eta \in C_0^\infty(H_{t'})$ such that $\eta = 1$ on H_t and $0 \leq \eta(x) \leq 1$ on $H_{t'}$ ([1], page 77).

Let $w(x) = v(x)\eta(x)$ defined on $H_{t'}$. We have

$$\begin{cases} w(x) = v(x) \text{ on } H_t \\ t - t' \leq w(x) \leq 0 \text{ on } H_{t'} \setminus H_t \\ w \in C_0^\infty(H_{t'}) \end{cases}.$$

Thus $w \leq 0$ on $H_{t'}$ and $w \in W_0^{1,2}(H_{t'})$, hence we can use Polya-Szego theorem as before, we get $w_* \in W^{1,2}(c, |H_{t'}|)$, $\forall c > 0$. Without difficulty, we can prove the following remark:

$$w_*|_{[0,|H_t|)} = v_*|^{(8)}_{[0,|H_t|)}.$$

From this we imply $v_* \in W^{1,2}(c, |H_t|), \forall c > 0$, which implies $v_* \in C^{0,1/2}[c, |H_t|]$. On the other hand, it is easy to prove that $v_*(s) = u_*(s) - t'^{(9)}$, this implies $u_* \in C^{0,1/2}[c, |H_t|]$ since $u_* \in C^{0,1/2}[0, |H_t|)$, moreover $u_* \in C^{0,1/2}[0, |\Omega|)$.$\square$

Because of this result, all the successive arguments until the estimate (4.4) are still true if we omit the right endpoints of the closed intervals.

We obtain the following theorem, which is an extension of Posteraro[6], theorem 2.1.

Theorem 4.1 *Suppose that u is a solution of* (4.5). *One has the estimate*

$$\min_{x\in\Omega} u(x) \geq \log(\frac{8\pi}{a|\Omega|}).$$

Proof. Since $\lim_{x\to\partial\Omega} u(x) = +\infty$

$$\lim_{s\to|\Omega|} \int_0^s e^{u_*(\tau)} d\tau \quad = \quad \lim_{t\to\infty} \int_0^{\mu(t)} e^{u_*(\tau)} d\tau$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \int_{H_t} e^{u(x)} dx \\
&= \int_{\Omega} e^{u(x)} dx = +\infty
\end{aligned}
$$

(in the above we have used lemma 3.2; see [6], page 447).

According to (4.4)

$$\int_0^s e^{u_*(\tau)} d\tau \leq \frac{\frac{8\pi}{a} s}{\frac{8\pi}{ae^M} - s}, \ \forall s \in [0, \frac{8\pi}{ae^M}).$$

Because of the above result we deduce that

$$|\Omega| \geq \frac{8\pi}{ae^M},$$

or

$$e^M \geq \frac{8\pi}{a|\Omega|},$$

Thus

$$\min_{x\in\Omega} u(x) = M \geq \log(\frac{8\pi}{a|\Omega|}).$$

We get the desired result.□

Appendix

In this section we state without proof several theorems we have used.

Theorem (Sard). *Given $u \in C^\infty(I\!R^n)$. The set of critical values of u is of (1-dimensional) measure zero.*

See ([1], page 34). A more general form of this theorem can be found in (S.Sternberg, *Lectures on Differential Geometry*, Prentice Hall, 1964, page 47).

Theorem (Federer). *Given $u : R^n \to I\!R$ is a Lipschitz function. Let $g : I\!R^n \to I\!R$ be an integrable function. Then*

$$\int_{I\!R^n} g(x)|\nabla u(x)|dx = \int_{I\!R} \int_{G_t} g(x) d\Gamma dt$$

where Γ is surface measure.

For this theorem, also see ([1], page 37).

Theorem (Sobolev Imbedding Theorem). *Let Ω be a bounded domain in $I\!R^n$.*

(i). If $n = 1$ then $W^{1,2}(\Omega) \hookrightarrow C^{0,1/2}(\bar{\Omega})$.

(ii). If $n = 2$ and $\partial\Omega \in C^{0,1}$ then $W^{k,2}(\Omega) \hookrightarrow C^{k-2}(\bar{\Omega})$, $\forall k \geq 2$.

See ([1], page 61; [2], page 151, corollary 7.11). Notice that if $\partial\Omega \in C^2$ then $\partial\Omega \in C^{0,1}$ ([2], page 89).

Footnotes

1. We need a lemma about computing the derivative of composite function.

LEMMA. Let $u \in W^{k,2}(\Omega)$, $k \geq 2$. Then e^u has weak derivative and

$$D_i(e^u) = e^u D_i u.$$

Proof. This proof is based on ([2], lemma 7.5). Since $\partial\Omega \in C^2$ we have $C^\infty(\bar{\Omega})$ is dense in $W^{k,2}(\Omega)$ (see [1], chapter 1.1.6). This imply $\exists \{u_n\} \subset C^\infty(\bar{\Omega})$ such that $u_n \xrightarrow{n\to\infty} u$ in $W^{k,2}(\Omega)$. Besides $W^{k,2}(\Omega) \hookrightarrow C^{k-2}(\bar{\Omega})$ (see the Appendix), this implies

$$||u_n - u||_{L^\infty(\Omega)} \leq ||u_n - u||_{W^{k,2}(\Omega)}.$$

Hence $||u_n - u||_{L^\infty(\Omega)} \xrightarrow{n\to\infty} 0$. We can assume $||u_n|| \leq c_1$ and $||u|| \leq c_1$.

We have

$$\begin{aligned} ||e^{u_n} - e^u||_{L^\infty(\Omega)} &\leq ||\sup_{\theta \in \bar{B}(0,c_1)} e^\theta (u_n(x) - u(x))||_{L^\infty(\Omega)} \\ &\leq \sup_{\theta \in \bar{B}(0,c_1)} e^\theta ||u_n(x) - u(x)||_{L^\infty(\Omega)} \\ &\leq c_2 ||u_n(x) - u(x)||_{L^\infty(\Omega)}. \end{aligned}$$

This implies $e^{u_n} \xrightarrow{n\to\infty} e^u$ in $L^\infty(\Omega)$.

On the other hand for all $\Omega' \subset\subset \Omega$ we have

$$\begin{aligned} \int_{\Omega'} |e^{u_n} D_i u_n - e^u D_i u| dx &\leq \int_{\Omega'} |(e^{u_n} - e^u) D_i u + (D_i u_n - D_i u) e^{u_n}| dx \\ &\leq \int_{\Omega'} |e^{u_n} - e^u||D_i u| + |D_i u_n - D_i u| e^{u_n} dx. \end{aligned}$$

We have

- $\int_{\Omega'} |D_i u_n - D_i u| e^{u_n} dx \leq \sup_{\bar{B}(0,c_1)} e^\theta \int_{\Omega'} |D_i u_n - D_i u| dx \xrightarrow{n\to\infty} 0.$

• $\displaystyle\int_{\Omega'} |e^{u_n} - e^u||D_i u|dx \leq c_2||u_n - u||_{L^\infty(\Omega')} \int_{\Omega'} |D_i u|dx \overset{n\to\infty}{\longrightarrow} 0.$

Thus we imply that $\displaystyle\int_{\Omega'} |e^{u_n} D_i u_n - e^u D_i u|dx \overset{n\to\infty}{\longrightarrow} 0.$

According to ([2], theorem 7.4) we get the desired result.□

2. If $K \in C^\infty(\bar{\Omega})$ and $u \in C^{k-2}(\bar{\Omega})$ then $Ke^u \in C^{k-2}(\bar{\Omega})$, hence $Ke^u D_i u \in W^{k-2,2}(\Omega)$.

3. *Proof.*

(i). Obvious.

(ii). We have $u_*(s) = \inf\{t \in I\!R : |H_t| > s\}$. Consider $s \in (0, |\Omega|)$. We only need to prove that if $s > 0$ then $\inf\{t \in I\!R : |H_t| > s\} \neq -\infty$, that is $\{t \in I\!R : |H_t| > s\}$ bounded from below. Suppose the contrary $\forall t, |H_t| > s$. Consider the sequence $t_n \searrow -\infty$. We have $\cap_{n=1}^{\infty} H_{t_n} = \emptyset$, thus $\lim_{n\to\infty} |H_{t_n}| = |\cap_{n=1}^{\infty} H_{t_n}| = 0$. On the other hand, $\lim_{n\to\infty} |H_{t_n}| \geq s > 0$. We get a contradiction.

(iii). Suppose that $s_1 < s_2$. Since $|H_t| > s_2 \Rightarrow |H_t| > s_1$ we have $\{t \in I\!R : |H_t| > s_2\} \subset \{t \in I\!R : |H_t| > s_1\}$, this implies $u_*(s_1) \leq u_*(s_2)$.

(iv). Let $t_0 = u_*(0) = \inf\{t \in I\!R : |H_t| > 0\}$. By definition, $ess \inf_\Omega u = \sup\{t \in I\!R : |H_t| = 0\}$.

• If $t_0 = -\infty$, meaning $\forall c \in I\!R, \exists t < c : |H_t| > 0$ then we have $\forall t \in I\!R, |H_t| > 0$, this implies $\{t \in I\!R : |H_t| = 0\} = \emptyset$, hence $ess \inf_\Omega u = -\infty$.

• If $t_0 > -\infty$, we have $\forall t < t_0, |H_t| = 0$ v $\forall t > t_0, |H_t| > 0$ (by definition inf, $\exists t_0 < \theta < t : |H_\theta| > 0$ which implies $|H_t| > 0$). This implies

$$\sup\{t \in I\!R : |H_t| = 0\} = \sup\{t \in I\!R, t < t_0\} = t_0.$$

4.

(i). We have $u_*(s) = \inf\{t \in I\!R, |H_t| > s\}$. If $u \leq v$ then $v(x) < t \Rightarrow u(x) < t$, this implies $\{x \in \Omega : v(x) < t\} \subset H_t$, thus

$$\begin{array}{ll} & |\{x \in \Omega : v(x) < t\}| > s \Rightarrow |H_t| > s \\ \Rightarrow & \{t \in I\!R, |\{x \in \Omega : v(x) < t\}| > s\} \subset \{t \in I\!R, |H_t| > s\} \\ \Rightarrow & u_*(s) \leq v_*(s). \end{array}$$

(ii). We have

$$\begin{aligned} (cu)_*(s) &= \inf\{t \in I\!R : |\{x \in \Omega : cu(x) < t\}| > s\} \\ &= \inf\{t \in I\!R : |\{x \in \Omega : u(x) < t/c\}| > s\} \\ &= \inf\{ct \in I\!R : |\{x \in \Omega : u(x) < t\}| > s\} \\ &= c\inf\{t \in I\!R : |\{x \in \Omega : u(x) < t\}| > s\} \\ &= cu_*(s). \end{aligned}$$

(iii). Notice that

$$e^{t_0} = \inf\{e^t, t > 0 : |H_t| > s\} \Rightarrow t_0 = \inf\{t \in I\!R : |H_t| > s\}.$$

Indeed, suppose that t_0 is defined by the left hand side. We have

$$t > 0 : |H_t| > s \Rightarrow e^t \geq e^{t_0} \Rightarrow t \geq t_0.$$

and

$$\begin{aligned} & \forall \epsilon > 0, \exists t, |H_t| > s : e^t < e^{t_0} + e^{t_0}\epsilon = e^{t_0}(1+\epsilon) < e^{t_0}e^{\epsilon} = e^{t_0+\epsilon} \\ \Rightarrow\ & \forall \epsilon > 0, \exists t, |H_t| > s : t < t_0 + \epsilon. \end{aligned}$$

Thus $t_0 = \inf\{t \in I\!R : |H_t| > s\}$.

We have

$$\begin{aligned} (e^u)_*(s) &= \inf\{t \in I\!R : |\{x \in \Omega : e^{u(x)} < t\}| > s\} \\ &= \inf\{t > 0 : |\{x \in \Omega : u(x) < \log t\}| > s\} \end{aligned}$$

$$
\begin{aligned}
&= \inf\{e^t, t > 0 : |\{x \in \Omega : u(x) < t\}| > s\} \\
&= e^{\inf\{t \in I\!R : |\{x \in \Omega : u(x) < t\}| > s\}} \\
&= e^{u_*(s)}.
\end{aligned}
$$

5. Since in $[\min_{\bar{\Omega}} u, \max_{\bar{\Omega}} u)$ the function μ is strictly increasing, we have $\forall t \in (\min_{\bar{\Omega}} u, \max_{\bar{\Omega}} u), \mu(t) > \mu(\min_{\bar{\Omega}} u) = |\{x \in \Omega : u(x) < \min_{\bar{\Omega}} u\}| = 0$.

6.

- We will prove $e^{u_*} \in C^{0,1/2}[0, |\Omega|]$. For $x > y \in [0, |\Omega|]$, we have

$$
|e^{u_*(x)} - e^{u_*(y)}| \le e^{\theta} |u_*(x) - u_*(y)|
$$

with $\theta \in [u_*(y), u_*(x)] \subset [\min_{\bar{\Omega}} u, \max_{\bar{\Omega}} u]$, hence $e^{\theta} \le c_1$. This implies the expression on the right hand side of the above inequality $\le c_2 |u_*(x) - u_*(y)| \le c_3 |x - y|^{1/2}$.

- If f and g are Hölder continuous of degree α on $[0, |\Omega|]$ then fg is Hölder continuous of degree α on $[0, |\Omega|]$. Indeed, $\forall x, y \in [0, |\Omega|]$,

$$
\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |(f(x) - f(y))g(x) + (g(x) - g(y))f(y)| \\
&\le |f(x) - f(y)||g(x)| + |g(x) - g(y)||f(y)| \\
&\le c_1 |x - y|^{\alpha} + c_2 |x - y|^{\alpha}.
\end{aligned}
$$

7. In detail we can do as follows.

Proof. We prove by induction.
When $k = 2$, since $u \in C^2(\Omega)$ we have $u \in W^{2,2}(\Omega')$.
Suppose the conclusion of the lemma is true for k and we have $u \in W^{k,2}(\Omega')$.
According to proposition 2.1 we have $Ke^u \in W^{k-1,2}(\Omega')$. According to lemma

2.1 we have $u \in W^{k+1,2}(\Omega^{'})$. So $u \in W^{k,2}(\Omega^{'})$, $\forall k \geq 2$. This implies, according to Sobolev Imbedding Theorem, $u \in C^{\infty}(\Omega^{'})$, $\forall \Omega^{'} \subset\subset \Omega$ (see the Appendix), this implies $u \in C^{\infty}(\Omega)$.□

8. *Proof.* . Consider $s \in [0, |H_t|)$.

$$w_*(s) = \inf\{\theta : |\{x \in H_{t'} : w(x) < \theta\}| > s\}.$$

Since

$$\begin{aligned} |\{x \in H_{t'} : w(x) < t - t^{'}\}| &= |\{x \in H_t : w(x) < t - t^{'}\}| \\ &= |H_t| > s, \end{aligned}$$

we have

$$\begin{aligned} & \inf\{\theta : |\{x \in H_{t'} : w(x) < \theta\}| > s\} \\ = \; & \inf\{\theta \leq t - t^{'} : |\{x \in H_{t'} : w(x) < \theta\}| > s\} \\ = \; & \inf\{\theta : |\{x \in H_t : w(x) < \theta\}| > s\} \\ = \; & \inf\{\theta : |\{x \in H_t : v(x) < \theta\}| > s\} \\ = \; & v_*(s), \end{aligned}$$

we get the desired result.□

9. We have

$$\begin{aligned} v_*(s) &= \inf\{\theta : |\{x \in \Omega : v(x) < \theta\}| > s\} \\ &= \inf\{\theta : |\{x \in \Omega : u(x) - t^{'} < \theta\}| > s\} \\ &= \inf\{\theta : |\{x \in \Omega : u(x) < t^{'} + \theta\}| > s\} \\ &= \inf\{\theta - t^{'} : |\{x \in \Omega : u(x) < \theta\}| > s\} \\ &= \inf\{\theta : |\{x \in \Omega : u(x) < \theta\}| > s\} - t^{'} \\ &= u_*(s) - t^{'}. \end{aligned}$$

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