

Exercises : Groups

1. Determine whether each of the following systems forms a group $(G, *)$, and which are abelian groups. Give reasons for your answers.

a) $G = \mathbb{Q} \setminus \{-6\}$, $x * y = 90xy + 540x + 540y + 3234$.

b) $G = \mathbb{R}^+$, $x * y = \ln(e^x + e^y - 1)$, where $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$.

c) $G = \mathbb{R}$, $x * y = x\sqrt{y^2 + 1} + y\sqrt{x^2 + 1}$.

d) $G = \mathbb{R} \times \mathbb{R}$, $(x_1, y_1) * (x_2, y_2) = (x_1 + x_2, e^{x_2} y_1 + y_2)$.

e) $G = \mathbb{R}^* \times \mathbb{R}$, $(x_1, y_1) * (x_2, y_2) = (x_1 x_2, x_1 y_2 + y_1)$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$

f) $G = \mathbb{R}^+ \setminus \{1\}$, $x * y = x^{\ln y}$.

g) $G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{Q}, a \neq 0 \right\}$, operation matrix multiplication.

2. Suppose G is a group such that $(ab)^2 = a^2 b^2$ for every $a, b \in G$. Show that G is abelian.

3. a) Show that a and a^{-1} are of the same order.

- b) Show that ab and ba are of the same order.

4. Let g be an element of order n in a group, and let $m \geq 1$. Show that

- a) If $\gcd(n, m) = d$, then g^m has order n/d .

- b) In particular, if m divides n , then g^m has order n/m .

5. Write the permutations as a product of disjoint cycles. Find the order of each permutation and state whether the permutation is even or odd

a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 3 & 4 & 5 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 1 & 5 & 7 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 4 & 2 & 7 & 3 & 5 & 1 & 6 \end{pmatrix}$

6. If a group G has only one element a of order 2, show that for every x in G , $xa = ax$.

7. Suppose $|G| = p$ where p is prime. Show that G is cyclic.

8. If G is a group in which $a^2 = e$, the identity for all $a \in G$, show that G is abelian.

9. If H and K are subgroups of a group G , prove that $H \cap K$ is also a subgroup of G . Is $H \cup K$ necessarily a subgroup of G ?
10. a) Show that $H = \left\{ \begin{pmatrix} x & y \\ 2y & x \end{pmatrix} \mid x, y \in \mathbb{Q} \right\}$ is a subgroup of $(M(2, \mathbb{Q}), +)$.
 b) Show that $H = \left\{ \begin{pmatrix} x & y \\ 2y & x \end{pmatrix} \mid x, y \in \mathbb{Q}; x^2 + y^2 > 0 \right\}$ is a subgroup of $(GL(2, \mathbb{Q}), \cdot)$.
11. Given $m, n \in \mathbb{Z}$. Prove that $m\mathbb{Z} \cap n\mathbb{Z} = \text{lcm}(m, n)\mathbb{Z}$ and $m\mathbb{Z} + n\mathbb{Z} = \text{gcd}(m, n)\mathbb{Z}$
12. a) Find all the group morphisms from $(\mathbb{Q}, +)$ to $(\mathbb{Z}, +)$.
 b) Find all the group morphisms from \mathbb{Z}_{12} to \mathbb{Z}_{12} .
13. a) Is $(\mathbb{Z}, +)$ isomorphic to (\mathbb{Q}^*, \cdot) , where $\mathbb{Q}^* = \mathbb{Q} - \{0\}$? Give reasons.
 b) Is $(\mathbb{R}, +)$ isomorphic to (\mathbb{R}^+, \cdot) , where $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$? Give reasons.
14. The **center** of a group G is the set $Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}$. Show that $Z(G)$ is an abelian subgroup of G .
15. In any group (G, \cdot) the element $a^{-1}b^{-1}ab$ is called the **commutator** of a and b . Let G' be the subset of G consisting of all finite products of commutators. Show that G' is a normal subgroup of G . This is called the **commutator subgroup**. Also prove that G/G' is abelian.
16. Let H be subgroup of G with only two right coset. Show that H is a normal subgroup of G .
17. Suppose H and N are subgroups of G with N normal. Prove that $H \cap N$ is normal in H and $H/(H \cap N)$ is isomorphic to HN/N .
18. Let \mathbb{C}^* be the group of nonzero complex numbers under multiplication and let W be the multiplicative group of complex numbers of unit modulus. Describe \mathbb{C}^*/W .
19. Show that $GL(n, \mathbb{R})/SL(n, \mathbb{R})$ is isomorphic to \mathbb{R}^* .
20. Show that every group of order ≤ 5 is abelian.
21. Show that cyclic groups of the same order are isomorphic.
22. Prove that if G and H are finite groups whose orders are relatively prime, there is only one morphism from G to H , the trivial one.
23. An automorphism of group G is isomorphism $G \rightarrow G$. For a fixed $g \in G$, the mapping $\gamma_g : G \rightarrow G$, defined by $\gamma_g(x) = gxg^{-1}$, is an automorphism.
 i) Prove that $\text{Aut}(G)$, the set of all the automorphism of a group G , is group under composition.
 ii) Prove that $\gamma : G \rightarrow \text{Aut}(G)$, defined by $g \rightarrow \gamma_g$, is a homomorphism.
 iii) Prove that $\ker \gamma = Z(G)$.
 iv) Prove that $\text{im } \gamma = \text{Aut}(G)$.