Exercises : Rings, Integral Domains, Fields.

1. Show that in a ring R with a unit element 1:
   i) a.0 = 0.a = 0.
   ii) a(- b) = (-a)b = -ab
   iii) (-a)(-b) = ab
   iv) (-1)a = -a ,
   v) (-1)(-1) = 1
   vi) (na)b = a(nb) = n(ab) ; na = (n1)a = a(n1)

   where a,b $\in$ R , n $\in \mathbb{Z}$
2. Let R be a ring . Suppose $a^2 = a$ for every a $\in$ R. Prove that R is a commutative ring.
3. Let R be ring with unit element 1 .We make R into another ring R' by defining
   $a \oplus b = a + b - 1$; a.b = ab +a + b.
   Verify that R' is a ring.
4. Let I and J be ideals in R . Prove that I + J = {x + y | x $\in$I , y $\in$J }and I $\cap$ J are also ideals in R.
5. Show that the only ideals in a field K are {0} and K.
6. A complex number a + bi where a, b are integers is called a Gaussian integer. Show that the set G of Gaussian integers is an integral domain.
7. Show that $S = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} | a, b \ reals \right\}$ is a subring of $M_2(\mathbb{R})$ isomorphic to $\mathbb{C}$.
8. i) Find all the ring morphism from $\mathbb{Q}$ to $\mathbb{Q}$.

   ii) Find all the ring morphism from $\mathbb{Q}\left(\sqrt{2}\right)$ to $\mathbb{Q}\left(\sqrt{2}\right)$.

   iii) Find all the ring morphism from $\mathbb{Q}$(i) to $\mathbb{Q}$(i).

9. Use the euclidean algorithm to find the gcd (f(x),g(x)) in $\mathbb{Q}[x]$,$\mathbb{Z}_5[x]$:
   i) $f(x) = 4x^4 - 2x^3 - 16x^2 + 5x + 9$; $g(x) = 2x^3 - x^2 - 5x + 4$.
   ii) $f(x) = x^5 + 3x^4 + x^3 + x^2 + 3x + 1$; $g(x) = x^4 + 2x^3 + x + 2$.
   iii) $f(x) = 4x^4 - 8x^3 + 9x^2 - 5x + 1$; $g(x) = 4x^4 + x^2 + 3x + 1$.
10. Solve the following equations:
    i) $21\bar{x} + \overline{24} = \overline{101}$ in $\mathbb{Z}_{103}$.

    ii) $68(\bar{x} + \overline{24}) = \overline{102}$ in $\mathbb{Z}_{492}$.

    iii) $78\bar{x} - \overline{13} = \overline{35}$ in $\mathbb{Z}_{666}$.

11. (i) Let ϕ : A $\rightarrow$ R be an isomorphism, and let ψ: R $\rightarrow$ A be its inverse. Show that ψ is an isomorphism.
    (ii) Show that the composite of two homomorphisms (isomorphisms) is again a homomorphism (isomorphism).

(iii) Show that $A \cong R$ defines an equivalence relation on the class of all commutative rings

12. Let R, R' be rings. Let $R \times R'$ be the set of all pairs (x, x') with $x \in R$ and $x' \in R'$. Show how one can make $R \times R'$ into a ring, by defining addition and multiplication componentwise. In particular, what is the unit element of $R \times R'$.
13. Let R be a ring, and Z the set of all elements $a \in R$ such that ax = xa for all $x \in R$. Show that Z is a subring of R, called the *center* of *R*.
14. Let R be a ring and $a \in R$. Let J be the set of all $x \in R$ such that xa = 0. Show that J is a left ideal.
15. An element *x* is called *nilpotent* if there exists a positive integer *n* such that $x^n = 0$.
    If R is commutative and x, y are nilpotent, show that x + y is nilpotent.
16. **Chinese Remainder Theorem.** Let R be a commutative ring and let I , J be ideals. They are called *relatively prime* if I + J = R**.** Suppose I and J are relatively prime. Given elements a, $b \in R$ show that there exists $x \in R$ such that $x \equiv a \pmod{I}$ and $x \equiv b \pmod{J}$ [This result applies in particular when $R = \mathbb{Z}$, $I = m\mathbb{Z}$ and $J = n\mathbb{Z}$ with relatively prime integers m, n.]
17. Let R be a commutative ring. An ideal P is said to be a *prime ideal* if $P \neq R$, and whenever a, $b \in R$ and $ab \in P$ then $a \in P$ or $b \in P$. Show that a non-zero ideal of $\mathbb{Z}$ is prime if and only if it is generated by a prime number.

19. Let R be a commutative ring. An ideal M of R is said to be a *maximal ideal* if $M \neq R$, and if there is no ideal J such that $R \supset J \supset M$ and $R \neq J$, $J \neq M$. Show that every maximal ideal is prime.

20. Let R be a commutative ring.

(a) Show that an ideal P is prime if and only if R/P is integral.

(b) Show that an ideal M is maximal if and only if R/M is a field.

21. Let K be a field and let $f : \mathbb{Z} \to K$ be the homomorphism of the integers into K. If the kernel of f is {0}, then K contains $\mathbb{Z}$ as a subring, and we say that K has *characteristic* 0 . If the kernel of f is generated by a prime number *p*, then we say that K has *characteristic p.*

Let K be a field of characteristic *p*. Show that $(x+y)^p = x^p + y^p$ for all x, $y \in K$.

22. Let K be a finite field of characteristic p. Show that the map $x \mapsto x^p$ is an automorphism of K.

23. Show that a ring-homomorphism of a field K into a ring $R \neq \{0\}$ is an isomorphism of K onto its image.