Sets
Set Operations
Functions

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1. Sets

1.1 Introduction and Notation
1.2 Cardinality
1.3 Power Set
1.4 Cartesian Products

1.1 Introduction and notation

. Introduction :A *set* is an unordered collection of elements.

Examples.

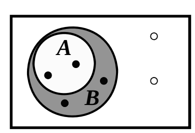
{1, 2, 3} is the set containing “1” and “2” and “3.”
{1, 1, 2, 3, 3} = {1, 2, 3} since repetition is irrelevant.
{1, 2, 3} = {3, 2, 1} since sets are unordered.
{0,1, 2, 3, …} is a way we denote an infinite set (in this case, the natural numbers).
∅ = {} is the empty set, or the set containing no element.

Note: ∅ ≠ {∅}

1.1 Definitions and notation

$x \in S$ means “x is an element of set S.”
$x \notin S$ means “x is not an element of set S.”
$A \subseteq B$ means “A is a subset of B.”

or, “B contains A.”
or, “every element of A is also in B.”
or, $\forall x\ ((x \in A) \rightarrow (x \in B))$.



Venn Diagram

1.1 Definitions and notation

$A \subseteq B$ means "A is a subset of B."
$A \supseteq B$ means "A is a superset of B."
$A = B$ if and only if A and B have exactly the same elements

iff, $A \subseteq B$ and $B \subseteq A$
iff, $A \subseteq B$ and $A \supseteq B$
iff, $\forall x\, ((x \in A) \leftrightarrow (x \in B))$.

So to show equality of sets A and B, show:

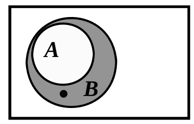
$A \subseteq B$

$B \subseteq A$

1.1 Definitions and notation

$A \subset B$ means "A is a proper subset of B."

- $A \subseteq B$, and $A \neq B$.
- $\forall x\, ((x \in A) \rightarrow (x \in B)) \wedge \exists x\, ((x \in B) \wedge (x \notin A))$



1.1 Definitions and notation

Quick examples:

- $\{1,2,3\} \subseteq \{1,2,3,4,5\}$
- $\{1,2,3\} \subset \{1,2,3,4,5\}$

Is $\varnothing \subseteq \{1,2,3\}$?

Yes! $\forall x\, (x \in \varnothing) \rightarrow (x \in \{1,2,3\})$ holds, because $(x \in \varnothing)$ is false.

Is $\varnothing \in \{1,2,3\}$? **No!**

Is $\varnothing \subseteq \{\varnothing,1,2,3\}$? **Yes!**

Is $\varnothing \in \{\varnothing,1,2,3\}$? **Yes!**

1.1 Definitions and notation

Quiz time:

Is $\{x\} \subseteq \{x\}$? Yes

Is $\{x\} \in \{x,\{x\}\}$? Yes

Is $\{x\} \subseteq \{x,\{x\}\}$? Yes

Is $\{x\} \in \{x\}$? No

Ways to define sets

- Explicitly: {John, Paul, George, Ringo}
- Implicitly: {1,2,3,…}, or {2,3,5,7,11,13,17,…}
- Set builder: $\{ x : x \text{ is prime} \}$, $\{ x \mid x \text{ is odd} \}$.
- In general $\{ x : P(x)\}$, where $P(x)$ is some predicate.

We read
"the set of all x such that $P(x)$"

Ways to define sets

In general $\{ x : P(x)\}$, where $P(x)$ is some predicate

Ex. Let $D(x,y)$ denote the predicate "x is divisible by y"

And $P(x)$ denote the predicate

$$\forall y\ ((y > 1) \wedge (y < x)) \rightarrow \neg D(x,y)$$

Then

$$\{ x : \forall y\ ((y > 1) \wedge (y < x)) \rightarrow \neg D(x,y) \}.$$

is precisely the set of all primes

1.2 Cardinality

If S is finite, then the ***cardinality*** of S, $|S|$, is the number of distinct elements in S.

If $S = \{1,2,3\}$ — $|S| = 3$.

If $S = \{3,3,3,3,3\}$ — $|S| = 1$.

If $S = \varnothing$ — $|S| = 0$.

If $S = \{ \varnothing, \{\varnothing\}, \{\varnothing,\{\varnothing\}\} \}$ — $|S| = 3$.

If $S = \{0,1,2,3,\ldots\}$, $|S|$ is infinite. (more on this later)

1.3 Power sets

If S is a set, then the ***power set*** of S is

$$P(S) = 2^S = \{ x : x \subseteq S \}.$$

We say, "$P(S)$ is the set of all subsets of S."

If $S = \{a\}$ — $2^S = \{\varnothing, \{a\}\}$.

If $S = \{a,b\}$ — $2^S = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$.

If $S = \varnothing$ — $2^S = \{\varnothing\}$.

If $S = \{\varnothing,\{\varnothing\}\}$ — $2^S = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing,\{\varnothing\}\}\}$.

Fact: if S is finite, $|2^S| = 2^{|S|}$. (if $|S| = n$, $|2^S| = 2^n$)

1.4 Cartesian Product

The ***Cartesian Product*** of two sets A and B is:

$$A \times B = \{ (a, b) : a \in A \wedge b \in B\}$$

If A = {Charlie, Lucy, Linus}, and
B = {Brown, VanPelt}, then

$A \times B$ = {(Charlie, Brown), (Lucy, Brown), (Linus, Brown), (Charlie, VanPelt), (Lucy, VanPelt), (Linus, VanPelt)}

We'll use these special sets soon!

$A_1 \times A_2 \times \ldots \times A_n =$
$= \{(a_1, a_2,\ldots, a_n): a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$

A,B finite $\rightarrow |A \times B| = |A||B|$

2. Set Operations

2.1 Introduction
2.2 Sets Identities
2.3 Generalized Set Operations
2.4 Computer Representation of Sets

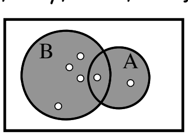
2.1 Introduction

The *union* of two sets A and B is:

$$A \cup B = \{ x : x \in A \vee x \in B\}$$

If A = {Charlie, Lucy, Linus}, and
B = {Lucy, Desi}, then

$A \cup B$ = {Charlie, Lucy, Linus, Desi}

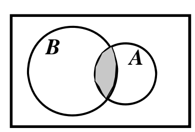


2.1 Introduction

The *intersection* of two sets A and B is:

$$A \cap B = \{ x : x \in A \wedge x \in B\}$$

If A = {Charlie, Lucy, Linus}, and
B = {Lucy, Desi}, then
$A \cap B$ = {Lucy}

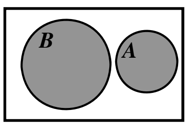


2.1 Introduction

The *intersection* of two sets A and B is:

$$A \cap B = \{ x : x \in A \wedge x \in B\}$$

If $A = \{x : x$ is a US president$\}$, and
$B = \{x : x$ is in this room$\}$, then
$A \cap B = \{x : x$ is a US president in this room$\} = \varnothing$



Sets whose intersection is empty are called *disjoint* sets

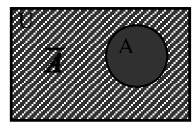
2.1 Introduction

The *complement* of a set A is:

$$\overline{A} = \{x : x \notin A\}$$

If $A = \{x : x$ is not shaded$\}$, then

$$\overline{A} = \{x : x \text{ is shaded}\}$$



$\overline{\varnothing} = U$
and
$\overline{U} = \varnothing$

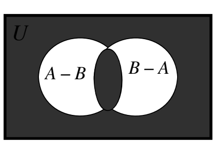
2.1 Introduction

The ***symmetric difference***, $A \oplus B$, is:

$$A \oplus B = \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

$$= (A - B) \cup (B - A)$$

$$= \{ x : x \in A \oplus x \in B\}$$



2.2 Set Identities

- *Identity* $A \cap U = A$, $A \cup \varnothing = A$
- *Domination* $A \cup U = U$, $A \cap \varnothing = \varnothing$
- *Idempotent* $A \cup A = A$, $A \cap A = A$

2.2 Set Identities

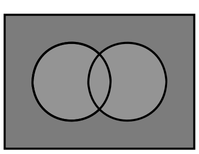
- *Excluded Middle* $A \cup \overline{A} = U$
- *Uniqueness* $A \cap \overline{A} = \emptyset$
- *Double complement* $\overline{\overline{A}} = A$

2.2 Set Identities

- *Commutativity* $A \cup B = B \cup A$
  $A \cap B = B \cap A$
- *Associativity* $(A \cup B) \cup C = A \cup (B \cup C)$
  $(A \cap B) \cap C = A \cap (B \cap C)$
- *Distributivity*
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2.2 Set Identities

- *DeMorgan's I* $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- *DeMorgan's II* $\overline{A \cap B} = \overline{A} \cup \overline{B}$



4 ways to prove identities

- Show that $A \subseteq B$ and that $A \supseteq B$. (New & important)
- Use a membership table. (Like truth tables)
- Use previously proven identities. (Like ≡)
- Use logical equivalences to prove equivalent set definitions. (Not hard, a little tedious)

4 ways to prove identities

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$$x \in \overline{A \cup B} \leftrightarrow x \notin A \cup B \leftrightarrow x \notin A \wedge x \notin B$$

$$\leftrightarrow x \in \overline{A} \wedge x \in \overline{B} \leftrightarrow x \in \overline{A} \cap \overline{B}$$

Not a particularly interesting example, sorry.

4 ways to prove identities

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

using a membership table.

0 : x is not in the specified set

1 : otherwise

Haven't we seen this before?

		$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$		$\overline{A \cup B}$

4 ways to prove identities

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

using logically equivalent set definitions

$\overline{(A \cup B)} = \{x : \neg(x \in A \vee x \in B)\}$

$= \{x : \neg(x \in A) \wedge \neg(x \in B)\}$

$= \{x : (x \in \overline{A}) \wedge (x \in \overline{B})\}$

$= \overline{A} \cap \overline{B}$

4 ways to prove identities

Prove that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

using known identities

$\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{(B \cap C)}$

$= \overline{A} \cap (\overline{B} \cup \overline{C})$

$= (\overline{B} \cup \overline{C}) \cap \overline{A}$

$= (\overline{C} \cup \overline{B}) \cap \overline{A}$

2.3 Generalized Set Operations

Generalized Union

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$= \{x : x \in A_1 \vee x \in A_2 \vee \ldots \vee x \in A_n\}$$

Ex. Let $U = \mathbf{N}$, and define:

$A_i=\{i, i+1, i+2, \ldots\}$

Then

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1,2,3,\ldots\}$$

Generalized Intersection

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

$$= \{x : x \in A_1 \wedge x \in A_2 \wedge \ldots \wedge x \in A_n\}$$

Ex. Let $U = \mathbf{N}$, and define:

$A_i=\{i, i+1, i+2, \ldots\}$

Then

$$\bigcap_{i=1}^{n} A_i = \{n, n+1, n+2, \ldots\}$$

2.4 Computer Representation of Sets

Let $U = \{x_1, x_2, \ldots, x_n\}$, and choose an arbitrary order of the elements of U, say

$x_1, x_2, \ldots, x_n$

Let $A \subseteq U$. Then the *bit string representation* of A is the bit string of length n : $a_1\ a_2 \ldots\ a_n$ such that $a_i = 1$ if $x_i \in A$, and 0 otherwise.

Ex. If $U = \{x_1, x_2, \ldots, x_6\}$, and $A = \{x_1, x_3, x_5, x_6\}$,

then the bit string representation of A is

(101011)

Sets as bit strings

Ex. If $U = \{x_1, x_2, \ldots, x_6\}$, $A = \{x_1, x_3, x_5, x_6\}$, and $B = \{x_2, x_3, x_6\}$.

Then we have a quick way of finding the bit string corresponding to of $A \cup B$ and $A \cap B$.

Bit-wise OR

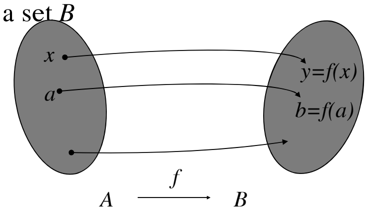
Bit-wise AND

3. Functions

3.1 Introduction
3.2 One-to-One and Onto Functions.
3.3 Inverse Functions and Composition of Functions
3.4 The Graphs of Functions
3.5 Some Important Functions

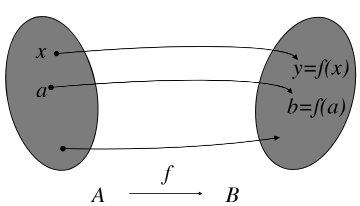
3.1 Introduction

Definition. A *function* (*mapping,map*) *f* is a rule that assigns to each element *x* in a set *A* exactly one element *y*=*f*(*x*) in a set *B*



- *A* is the ***domain***, *B* is the ***codomain*** of *f*.

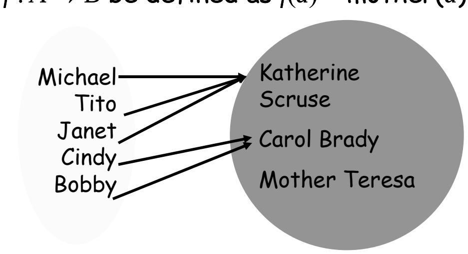
3.1 Introduction



- $b = f(a)$ is the ***image*** of *a* and *a* is the ***preimage*** of *b*.
- The ***range*** of *f* is the set $\{f(a),\ a \in A\}$

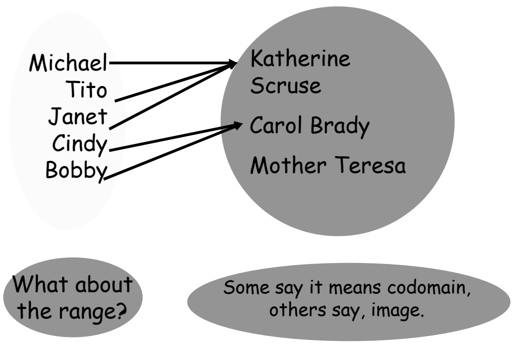
Example.

A = {Michael, Tito, Janet, Cindy, Bobby}
B = {Katherine Scruse, Carol Brady, Mother Teresa}
Let $f : A \to B$ be defined as $f(a) = $ mother(a).



For any set $S \subseteq A$, image(S) = $\{b : \exists a \in S, f(a) = b\} = f(S)$

So, image({Michael, Tito}) = {Katherine Scruse}
image(A) = B – {Mother Teresa}



Algebra of functions: let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg and f/g are defined as follows:

$$(f+g)(x) = f(x) + g(x) \quad \text{domain} = A \cap B$$

$$(f-g)(x) = f(x) - g(x) \quad \text{domain} = A \cap B$$

$$(fg)(x) = f(x)g(x) \quad \text{domain} = A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{domain} = \{x \in A \cap B \,/\, g(x) \neq 0\}$$

Example: let $f(x) = x^2$, $f(x) = x - x^2$ be functions from **R** to **R**, find $f + g$ and fg.

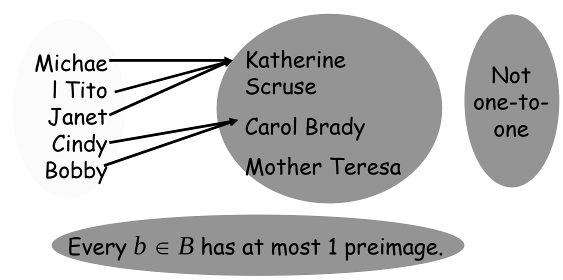
Solution: We have

$(f + g)(x) = x^2 + (x - x^2) = x$
And
$(fg)(x) = x^2 (x - x^2) = x^3 - x^4$

3.2 One-to-One and Onto Functions

Definition. A function f: $A \rightarrow B$ is *one-to-one* (*injective, an injection*) if

$$\forall x, y \, (f(x) = f(y) \rightarrow x = y)$$



Remark. A function f: A → B is *one-to-one* iff

$\forall x,y\ (x \neq y \rightarrow f(x) \neq f(y))$

Recall that

- ❖ A function *f* is **strictly increasing** on an interval $I \subseteq \mathbf{R}$ if

  $\forall x,y\ (x < y \rightarrow f(x) < f(y))$

- ❖ *f* is strictly **decreasing** on *I* if

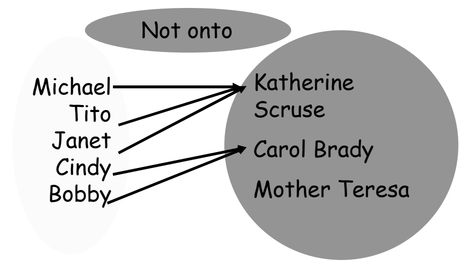
  $\forall x,y\ (x < y \rightarrow f(x) > f(y))$

- ❖ It is clear that a strictly increasing or strictly decreasing function is one-to-one.

Onto Functions

Definition. A function $f: A \rightarrow B$ is *onto* (*surjective*, a *surjection*) if $\forall b \in B, \exists a \in A\ f(a) = b$

Every $b \in B$ has at least 1 preimage.



Onto Functions

Example. Is the function $f(x) = x^2$ from **Z** to **Z** onto?

Solution: The function *f* is not onto since there is no *x* in **Z** such that $x^2 = -1$

Example. Is the function $f(x) = x + 1$ from **Z** to **Z** onto?

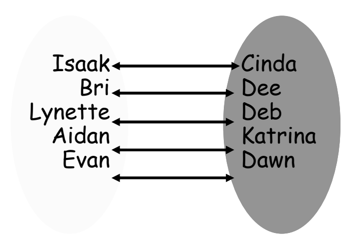
Solution: The function *f* is onto since for every

y in **Z,** there is an element *x* in **Z** such that $x + 1 = y$ (by taking $x = y - 1$)

Bijection

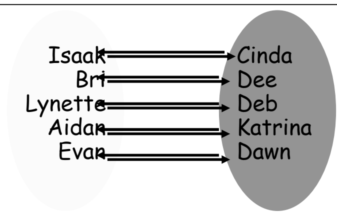
Definition. A function $f: A \rightarrow B$ is bijective if it is one-to-one and onto.

Every b ∈ B has exactly 1 preimage.



3.3 Inverse Functions and Compositions of Functions

Definition. Let $f : A \to B$ be a bijection. Then the inverse function of f, denoted by f^{-1} is the function that assigns each element b in B the unique element a in A such that $f(a) = b$. Thus $f^{-1}(b) = a$.



f^{-1}(Cinda) = Isaak, f^{-1}(Dee) = *Bri*, …, f^{-1}(Dawn) = Evan

Example. Is the function $f(x) = x^2$ from **Z** to **Z** invertible? (i.e. the inverse function exists)

Solution: The function f is not onto. Therefore it is not a bijection, and hence not invertible

Example. Is the function $f(x) = x + 1$ from **Z** to **Z** invertible?

Solution: The function f is a bijection so it is invertible.

Example. Is the function $f(x) = x + 1$ from **Z** to **Z** invertible? What is its inverse?

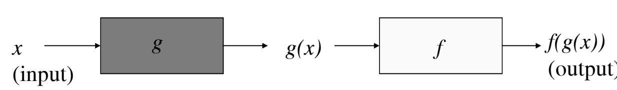
Solution: The function f is a bijection so it is invertible.
To find the inverse, let y be any element in **Z,** we find the element x in **Z** such that $y = f(x) = x + 1$.
Solving this equation we obtain $x = y - 1$.
Hence $f^{-1}(y) = y - 1$.

We also write $f^{-1}(x) = x - 1$.

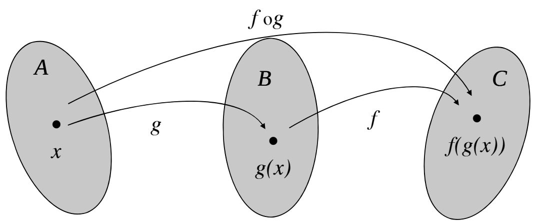
Definition. The composition of a function $g: A \to B$ and a function $f : B \to C$ is the function $f \circ g : A \to C$ defined by

$$f \circ g(x) = f(g(x))$$

Note. The domain of $f \circ g$ is also the domain of g, and the codomain of $f \circ g$ is also the codomain of f.



Arrow diagram for $f \circ g$



Example. let $f(x) = x^2$ and $g(x) = x - 3$ are functions from **R** to **R**.

Find the compositions $f \circ g$ and $g \circ f$

Solution.

$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$

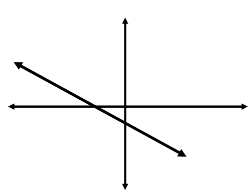
$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$

This shows that in general: $f \circ g \neq g \circ f$

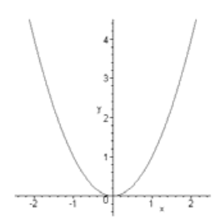
3.4 The Graph of a Function

Definition. Let $f : A \to B$ be a function. Then the *graph* of f, is the set of ordered pair (a, b) with a in A and $f(a) = b$.

Example. The graph of the function $f : \mathbf{R} \to \mathbf{R}$ such that $f(x) = -x/2 - 25$



Example. The graph of the function $f : \mathbf{R} \to \mathbf{R}$ such that $f(x)=x^2$



Example. The average CO_2 level in the atmosphere is a function of time given by the following table

The graph of this function

ppm:parts per million

3.5 Some Important Functions

Ceiling.
$f(x) = \lceil x \rceil$ the least integer y so that $x \le y$.

Ex: $\lceil 1.2 \rceil = 2; \lceil -1.2 \rceil = -1; \lceil 1 \rceil = 1$

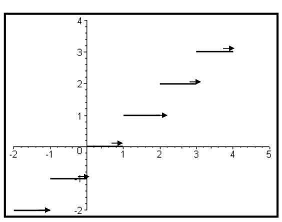
Floor.
$f(x) = \lfloor x \rfloor$ the greatest integer y so that $y \le x$.

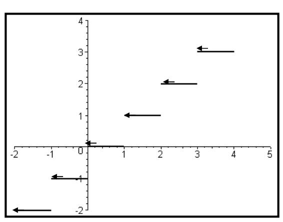
Ex: $\lfloor 1.8 \rfloor = 1; \lfloor -1.8 \rfloor = -2; \lfloor -5 \rfloor = -5$

Quiz: what is $\lceil -1.2 + \lfloor 1.1 \rfloor \rceil$?

0

The graph of the floor function





The graph of the ceiling function