

II

RELATIONS

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Relations

1. Định nghĩa và tính chất
2. Biểu diễn quan hệ
3. Quan hệ tương đương. Đồng dư. Phép toán số học trên \mathbb{Z}_n
4. Quan hệ thứ tự. Hasse Diagram

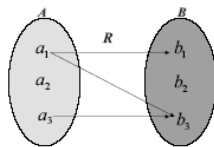
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1. Definitions

Definition. A quan hệ hai ngôi từ tập A đến tập B là tập con của tích Descartes $R \subseteq A \times B$.

Chúng ta sẽ viết $a R b$ thay cho $(a, b) \in R$

Quan hệ từ A đến chính nó được gọi là quan hệ trên A



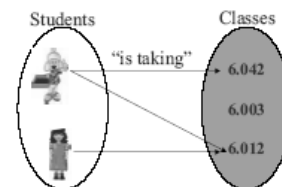
$$R = \{ (a_1, b_1), (a_2, b_3), (a_3, b_3) \}$$

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1. Definitions

Example. $A = \text{students}$; $B = \text{courses}$.

$$R = \{ (a, b) \mid \text{student } a \text{ is enrolled in class } b \}$$



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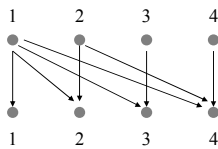
1. Definitions

Example. Let $A = \{1, 2, 3, 4\}$, and

$$R = \{(a, b) \mid a \text{ divides } b\}$$

Then R consists of the pairs:

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$$



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2. Properties of Relations

Definition. A relation R on a set A is *reflexive* (phản xạ) if:

$$(a, a) \in R \text{ for all } a \in A$$

Example. On the set $A = \{1, 2, 3, 4\}$, the relation:

■ $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
is not reflexive since $(3,3) \notin R_1$

■ $R_2 = \{(1,1), (1,2), (1,4), (2,2), (3,3), (4,1), (4,4)\}$
is reflexive since $(1,1), (2,2), (3,3), (4,4) \in R_2$

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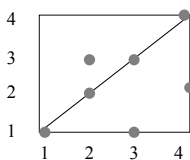
■ The relation \leq on \mathbf{Z} is reflexive since $a \leq a$ for all $a \in \mathbf{Z}$

■ The relation $>$ on \mathbf{Z} is not reflexive since $1 \not> 1$

■ The relation “ \mid ” (“divides”) on \mathbf{Z}^+ is reflexive since any integer a divides itself

Note. A relation R on a set A is reflexive iff it contains the diagonal of $A \times A$:

$$\Delta = \{(a, a); a \in A\}$$



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2. Properties of Relations

Definition. A relation R on a set A is *symmetric* (đối xứng) if:

$$\forall a \in A \forall b \in A (a R b) \rightarrow (b R a)$$

The relation R is said to be *antisymmetric* (phân xứng) if:

$$\forall a \in A \forall b \in A (a R b) \wedge (b R a) \rightarrow (a = b)$$

Example.

■ The relation $R_1 = \{(1,1), (1,2), (2,1)\}$ on the set $A = \{1, 2, 3, 4\}$ is symmetric

■ The relation \leq on \mathbf{Z} is not symmetric.

■ However it is antisymmetric since

$$(a \leq b) \wedge (b \leq a) \rightarrow (a = b)$$

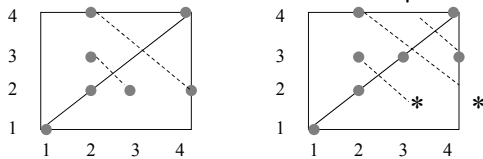
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- The relation “ $|$ ” (“divides”) on \mathbb{Z}^+ is not symmetric. However it is antisymmetric since

$$(a | b) \wedge (b | a) \rightarrow (a = b)$$

Note. A relation R on a set A is symmetric iff it is self symmetric with respect to the diagonal Δ of $A \times A$.

The relation R is antisymmetric iff the only self symmetric parts lie on the diagonal Δ of $A \times A$.



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2. Properties of Relations

Definition. A relation R on a set A is *transitive*(*bắc cầu, truyền*) if:

$$\forall a \in A \forall b \in A \forall c \in A (a R b) \wedge (b R c) \rightarrow (a R c)$$

Example.

- The relation $R = \{(1,1), (1,2), (2,1), (2, 2), (1, 3), (2, 3)\}$ on the set $A = \{1, 2, 3, 4\}$ is transitive
- The relations \leq and “ $|$ ” on \mathbb{Z} are transitive

$$(a \leq b) \wedge (b \leq c) \rightarrow (a \leq c)$$

$$(a | b) \wedge (b | c) \rightarrow (a | c)$$

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3. Representing Relations

Introduction
Matrices
Representing Relations

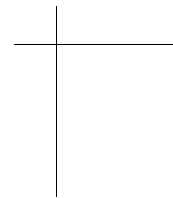
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Introduction

Let R be a relation from $A = \{1,2,3,4\}$ to $B = \{u,v,w\}$:

$$R = \{(1,u),(1,v),(2,w),(3,w),(4,u)\}.$$

Then we can represent R as:



The labels on the outside are for clarity. It's really the matrix in the middle that's important.

This is a 4×3 -matrix whose entries indicate membership in R

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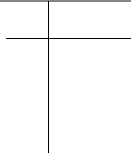
Representing Relations

Definition. Let R be a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$, then the **representing matrix** of R is the $m \times n$ zero-one matrix $\mathbf{M}_R = [m_{ij}]$ defined by

$$m_{ij} = \begin{cases} 0 & \text{if } (a_i, b_j) \notin R \\ 1 & \text{if } (a_i, b_j) \in R \end{cases}$$

Example. Let R be the relation from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$ such that $a R b$ if $a > b$.

Then the representing matrix of R is



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$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Example. Let R be the relation from $A = \{a_1, a_2, a_3\}$ to $B = \{b_1, b_2, b_3, b_4, b_5\}$ represented by the matrix

$$\mathbf{M}_R = \begin{array}{ccccc|l} & b_1 & b_2 & b_3 & b_4 & b_5 & \\ \hline & 0 & 1 & 0 & 0 & 0 & a_1 \\ & 1 & 0 & 1 & 1 & 0 & a_2 \\ & 1 & 0 & 1 & 0 & 1 & a_3 \end{array}$$

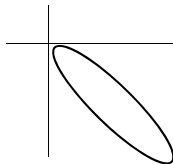
Then R consists of the pairs:

$\{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$

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Representing Relations

- Let R be a relation on a set A , then the matrix \mathbf{M}_R that represents R is a **square matrix**
- R is **reflexive** if and only if all **diagonal entries** of \mathbf{M}_R are equal to 1: $m_{ii} = 1$ for all i

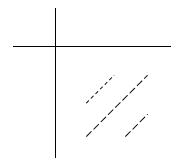


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Representing Relations

- Let R be a relation on a set A , then the matrix \mathbf{M}_R that represents R is a **square matrix**
- R is symmetric if and only if \mathbf{M}_R is **symmetric**

$$m_{ij} = m_{ji} \quad \text{for all } i, j$$

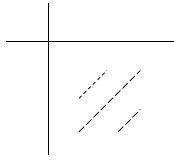


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Representing Relations

- Let R be a relation on a set \mathcal{A} , then the matrix \mathbf{M}_R that represents R is a *square matrix*
- R is *antisymmetric* if and only if \mathbf{M}_R satisfies:

$$m_{ij} = 0 \text{ or } m_{ji} = 0 \text{ if } i \neq j$$



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4. Equivalence Relations

Introduction
 Equivalence Relations
 Representation of Integers
 Equivalence Classes
 Linear Congruences.

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Introduction

- Example:
 Let $S = \{\text{people in this classroom}\}$, and let
 $R = \{(a,b): a\text{'s last name starts with the same letter as } b\text{'s last name}\}$

- Quiz time:

Is R reflexive?

Yes

Is R symmetric?

Yes

Is R transitive?

Yes

Everyone whose last name starts with the same letter as yours belongs to your assignment group.

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Equivalence Relations Quan hệ tương đương

Definition. A relation R on a set \mathcal{A} is an *equivalence relation* if it is reflexive, symmetric and transitive:

Example. Let R be the relation on the set of strings of English letters such that aRb if and only if a and b have the same length, then R is an equivalence relation

Example. Let R be the relation on \mathbf{R} such that aRb if and only if $a - b$ is an integer, then R is an equivalence relation

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Recall that if a and b are integers, then a is said to be divisible by b , or a is a multiple of b , or b is a divisor of a if there exists an integer k such that $a = kb$

Example. Let m be a positive integer and R the relation on \mathbf{Z} such that aRb if and only if $a - b$ is divisible by m , then R is an equivalence relation

■ The relation is clearly reflexive and symmetric.

■ Let a, b, c be integers such that $a - b$ and $b - c$ are both divisible by m , then $a - c = a - b + b - c$ is also divisible by m . Therefore R is transitive

■ This relation is called the ***congruence modulo m*** and we write

$$a \equiv b \pmod{m}$$

instead of aRb

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Equivalence Classes Lớp tương đương

Definition. Let R be an equivalence relation on a set A , and $a \in A$. The ***equivalence class of a*** denoted by $[a]_R$ or simply $[a]$ is the subset

$$[a]_R = \{b \in A, b R a\}$$

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Equivalence Classes

Example. What are the equivalence classes modulo 8 of 0 and 1?

Solution. The equivalence class modulo 8 of 0 contains all integer a with the same remainder mod 8 as 0, i.e. a is a multiple of 8. Therefore

$$[0]_8 = \{ \dots, -16, -8, 0, 8, 16, \dots \}$$

Similarly

$$\begin{aligned} [1]_8 &= \{a, a \text{ has remainder } 1 \pmod{8}\} \\ &= \{ \dots, -15, -7, 1, 9, 17, \dots \} \end{aligned}$$

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Note. In the last example, the equivalence classes $[0]_8$ and $[1]_8$ are disjoint.

More generally, we have

Theorem. Let R be an equivalence relation on a set \mathcal{A} and $a, b \in \mathcal{A}$, then

- (i) $a R b$ if and only if $[a]_R = [b]_R$
- (ii) $[a]_R \neq [b]_R$ if and only if $[a]_R \cap [b]_R = \emptyset$

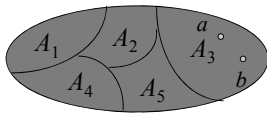
Note. The equivalence classes form a partition of the set \mathcal{A} in the sense that it divides \mathcal{A} into disjoint subsets.

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Note. Let $\{A_1, A_2, \dots\}$ be a partition of A into disjoint nonempty subsets then there is a unique equivalence relation R on A such that the given sets A_i are precisely the equivalence classes.

Let indeed $a, b \in A$, then we define $a R b$ if and only if there is a subset A_i such that $a, b \in A_i$

We can prove that R is an equivalence relation on A and $[a]_R = A_i$ if and only if $a \in A_i$



Example. Let m be a positive integer, then there are m different congruence classes $[0]_m, [1]_m, \dots, [m-1]_m$.

They form a partition of \mathbf{Z} into disjoint subsets.

■ Note that

$$\begin{aligned}
 [0]_m &= [m]_m = [2m]_m = \dots \\
 [1]_m &= [m+1]_m = [2m+1]_m = \dots \\
 [2]_m &= [m+2]_m = [2m+2]_m = \dots \\
 &\dots\dots\dots \\
 [m-1]_m &= [2m-1]_m = [3m-1]_m = \dots
 \end{aligned}$$

■ They are called the *integers modulo m*

■ The set of all integers modulo m is denoted by \mathbf{Z}_m

$$\mathbf{Z}_m = \{[0]_m, [1]_m, \dots, [m-1]_m\}$$

6. Partial Orderings

- Introduction
- Lexicographic Order
- Hasse Diagrams
- Maximal and Minimal Elements
- Upper Bounds and Lower Bounds
- Topological Sorting

Introduction

Example Let R be the relation on the real numbers:

$$a R b \text{ if and only if } a \leq b$$

Quiz time:

- Is R reflexive? Yes
- Is R transitive? Yes
- Is R symmetric? No
- Is R antisymmetric? Yes

Introduction

Definition. A relation R on a set A is a *partial order* (quan hệ thứ tự, thứ tự) if it is *reflexive*, *antisymmetric* and *transitive*.

We often denote a partial order by $<$

The pair $(A, <)$ is called a *partially ordered set* (tập sắp thứ tự) or a *poset*

Reflexive: $a < a$

Antisymmetric: $(a < b) \wedge (b < a) \rightarrow (a = b)$

Transitive: $(a < b) \wedge (b < c) \rightarrow (a < c)$

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Introduction

Definition. A relation R on a set A is a *partial order* if it is reflexive, antisymmetric and transitive.

Example. The divisibility relation “ $|$ ” on the set of positive integers is a partial ordering, i.e. $(\mathbb{Z}^+, |)$ is a poset

Reflexive? Yes, $x | x$ since $x = 1 \cdot x$

Transitive? Yes?

$a | b$ means $b = ka$, $b | c$ means $c = jb$.
Then $c = j(ka) = jka$: $a | c$

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Example. The divisibility relation “ $|$ ” on the set of positive integers is a partial ordering, i.e. $(\mathbb{Z}^+, |)$ is a poset

Antisymmetric? Yes?

$a | b$ means $b = ka$, $b | a$ means $a = jb$.
Then $a = jka$
It follows that $j = k = 1$, i.e. $a = b$

Example. Is $(\mathbb{Z}, |)$ a poset?

Antisymmetric?

No

$3|-3$, and $-3|3$,
but $3 \neq -3$.

Not a poset.

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Ex. Is $(2^S, \subseteq)$, where 2^S the set of all subsets of S , a poset?

Reflexive? Yes, $A \subseteq A$.

Transitive? Yes, $A \subseteq A, \forall A \in 2^S$

$A \subseteq B, B \subseteq C$. Does that mean $A \subseteq C$?

Antisymmetric?

Yes

$A \subseteq B, B \subseteq A$. Does that mean $A = B$?

Yes

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Definition. The elements a and b of a poset $(S, <)$ are *comparable* if either $a < b$ or $b < a$.

Otherwise, they are said to be *incomparable* (không so sánh được).

A poset $(S, <)$ such that every two elements are comparable is called a *totally ordered set* (tập sắp thứ tự toàn phần)

We also say that $<$ is a *total order* (thứ tự toàn phần) or a *linear order* (thứ tự tuyến tính) on S

Example. The relation “ \leq ” on the set of positive integers is a total order.

Example. The divisibility relation “ $|$ ” on the set of positive integers is not a total order, since the elements 5 and 7 are not comparable

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Lexicographic Order

Thứ tự tự điển

Ex. A straight forward partial order on bit strings of length n , is defined as:

$$a_1a_2\dots a_n \leq b_1b_2\dots b_n$$

if and only if $a_i \leq b_i, \forall i$.

With respect to this order, 0110 and 1000 are “incomparable” ...

We can't tell which is “bigger.”

For many applications in computer, it is convenient to have a total order on bit strings, or more generally on strings of characters:

This is the lexicographic order

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Lexicographic Order

Let (A, \leq) and (B, \leq') be two totally ordered sets. We define a partial order $<$ on $A \times B$ as follows:

$(a_1, b_1) < (a_2, b_2)$ if and only if

$a_1 < a_2$ or $(a_1 = a_2$ and $b_1 \leq' b_2)$

Now we can verify that this is a total order on $A \times B$ called the **lexicographic order**

Note that if A and B are well ordered by \leq and \leq' respectively, then $A \times B$ is also well ordered by $<$

Note also that this definition can be extended to the cartesian product of a finite number of totally ordered sets

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Lexicographic Order

Recall that if Σ is a finite set called an alphabet, then the set of strings on Σ , denoted by Σ^* is defined by:

- $\lambda \in \Sigma^*$, where λ denotes the null or empty string.
- If $x \in \Sigma$, and $w \in \Sigma^*$, then $wx \in \Sigma^*$, where wx is the concatenation of string w with symbol x .

Example. Let $\Sigma = \{a, b, c\}$. Then

$\Sigma^* = \{\lambda, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, aab, \dots\}$

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Lexicographic Order

Now assume that \leq is a total order on Σ , then we can define a total order $<$ on Σ^* as follows.

Let $s = a_1 a_2 \dots a_m$ and $t = b_1 b_2 \dots b_n$ be two strings in Σ^*

Then $s < t$ if and only if

- either $a_i = b_i$ for $1 \leq i \leq m$ so that

$$t = a_1 a_2 \dots a_m b_{m+1} b_{m+2} \dots b_n$$
- or there exists $k < m$ such that
 - ✓ $a_i = b_i$ for $1 \leq i \leq k$ and
 - ✓ $a_{k+1} < b_{k+1}$ so that

$$s = a_1 a_2 \dots a_k a_{k+1} a_{k+2} \dots a_m$$

$$t = a_1 a_2 \dots a_k b_{k+1} b_{k+2} \dots b_n$$

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- We can prove again that $<$ is a total order on the set Σ^* called the *lexicographic order* on Σ^*

Example. If Σ is the English alphabet with the usual order on the characters: $a < b < \dots < z$, then the lexicographic order is precisely the order of the words in a dictionary

For example

✓ $discreet < discrete$ $discreet$
 $\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow$ $discrete$ $e \not< t$

✓ $discreet < discreetness$ $discreet$
 $\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow$ $discreetness$

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$<$ is a total order called the *lexicographic order* on Σ^*

Example. If $\Sigma = \{0, 1\}$ with the usual order $0 < 1$, then Σ^* is the set of all bit strings.

We have

- ✓ $0110 < 10$
- ✓ $0110 < 01100$

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Hasse Diagrams

A poset can be represented visually using a special kind of graphs called the *Hasse diagram*

To define the Hasse diagram we need the concept of direct upper bound.

Definition. An element b in a poset $(S, <)$ is said to be an *upper bound* of an element a in S if $a < b$

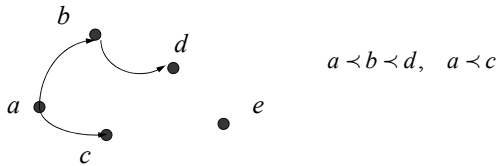
We also say that a is a *lower bound* of b
 b is said to be a *direct upper bound* of a if b is an upper bound of a , and there is no upper bound c such that

$$a < c < b, \quad a \neq c \neq b$$

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Hasse Diagrams

- Now the **Hasse diagram** of a finite poset $(S, <)$ is the graph:
 - whose vertices are points in the plane in one-to-one correspondence with S ,
 - two vertices a, b are joined by an arc directed from a to b if b is a direct upper bound of a



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Hasse Diagrams

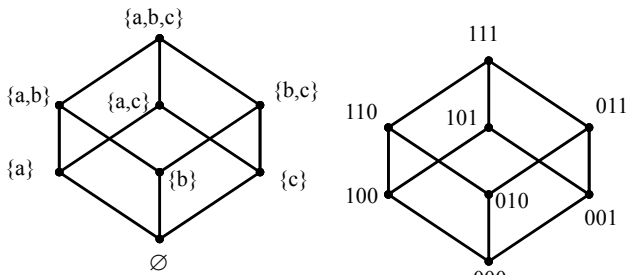
Ex. The Hasse diagram of the poset $(\{1,2,3,4\}, \leq)$ can be drawn as



Note. We did not draw up arrows for the arcs by adopting the convention that arcs are always directed upward

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Example. The Hasse diagram of $P(\{a,b,c\})$ and the Hasse diagram of the set of bit strings of length 3 with natural bitwise order



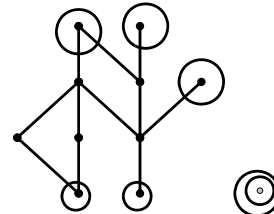
They look similar !!!

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Maximal & Minimal Elements

Consider this poset:

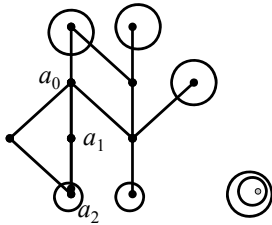
- Each Red is **maximal**: there is no proper upper bound
- Each Green is **minimal**: there is no proper lower bound
- There is no arc starting from a maximal element
- There is no arc ending at a minimal element



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Note. In a finite poset S , maximal and minimal elements always exist.

- ✓ In fact, we can start from any element $a_0 \in S$.
If a_0 was not minimal, then there exists $a_1 < a_0$,
and so on until a minimal element is found.
- ✓ The maximal elements are found in a similar way.



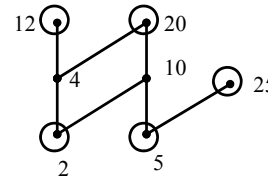
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Example. What are the maximal and minimal elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$?

Solution. From the Hasse diagram, we see that 12, 20, 25 are maximal elements

and 2, 5 are minimal elements

Thus the maximal and minimal elements of a poset are not necessarily unique



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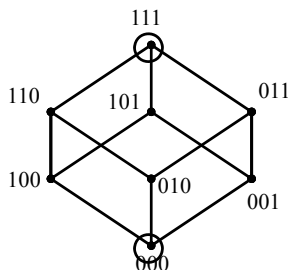
Example. What are the maximal and minimal elements of the poset consisting of bit strings of length 3?

Solution. From the Hasse diagram, we see that 111 is the unique maximal element and 000 is the unique minimal element

111 is also the *greatest element* and 000 is the *least element* in the sense:

$$000 < abc < 111$$

for all string abc



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In fact we have

Theorem. In a finite poset, if the maximal element is unique, then it is the greatest element. Similarly for the least element.

Proof. Let g be the unique maximal element.

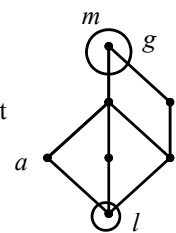
Let a be an arbitrary element, then there is a maximal element m such that

$$a < m$$

Since g is unique we must have $m = g$, i.e. $a < g$

Therefore g is the greatest element.

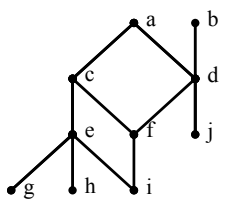
Similar proof for the existence of the least element l



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Upper and Lower Bounds

Definition. Let (S, \succ) be a partial order. If $A \subseteq S$, then an **upper bound** for A is an element $x \in S$ (perhaps in A also) such that $\forall a \in A, a \prec x$.
 A **lower bound** for A is an $x \in S$ such that $\forall a \in A, x \prec a$

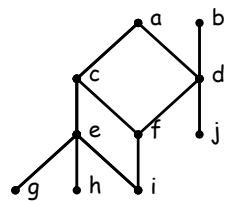


Ex. The upper bound of $\{g, j\}$ is a .
 Why not b ?

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Upper and Lower Bounds

Definition. Let (S, \prec) be a partial order. If $A \subseteq S$, then an **upper bound** for A is an element $x \in S$ (perhaps in A also) such that $\forall a \in A, a \prec x$.
 A **lower bound** for A is an $x \in S$ such that $\forall a \in A, x \prec a$



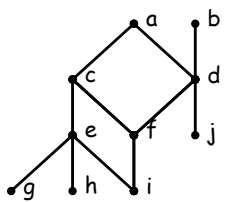
Ex. The upper bounds of $\{g, i\}$ are
 (A) e
 (B) c
 (C) $e, c,$ and a

$\{a, b\}$ has no UB.

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Upper and Lower Bounds

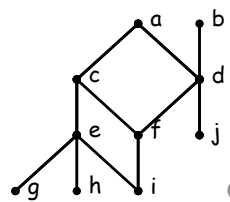
Definition. Let (S, \prec) be a partial order. If $A \subseteq S$, then an **upper bound** for A is an element $x \in S$ (perhaps in A also) such that $\forall a \in A, a \prec x$.
 A **lower bound** for A is an $x \in S$ such that $\forall a \in A, x \prec a$



Ex. The lower bounds of $\{c, d\}$ are
 (A) f
 (B) f, i
 (C) e
 NO! e, d are not comparable
 $\{g, h\}$ has no LB.

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Definition. Let (S, \prec) be a partial order. If $A \subseteq S$, then the **least upper bound** for A is an upper bound x such that for any upper bound y of $A, y \succ x$.
 The **greatest lower bound** for A is a lower bound x such that for any lower bound y of $A, y \prec x$



Ex. The LUB of $\{i, j\}$ is d
 Ex. The GLB of $\{g, j\}$ is
 (A) e, c, a NO! they are upper bounds
 (B) d, a NO! they are upper bounds
 $\{g, j\}$ has no lower bounds, and hence no GLB

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If the least upper bound of $A = \{a, b\}$ exists, then we denote it by $a \vee b$

Similarly if the greatest lower bound of $A = \{a, b\}$ exists, then we denote it by $a \wedge b$

Ex. $i \vee j = d$

Ex. $b \wedge c = f$

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Topological Sorting

Consider the problem of getting dressed.

Precedence constraints are modeled by a poset in which $a < b$ if and only if you must put on a before b .

In what order will you get dressed while respecting constraints?

In other words, we will find a new total order so that a is a lower bound of b if $a < b$

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Topological Sorting

Recall that every finite non-empty poset has at least one minimal element a_1 .

E.g. shirt is a minimal element

✓ Now the new set after we remove a_1 is still a poset.

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Topological Sorting

✓ Let a_2 be a minimal of the new poset.

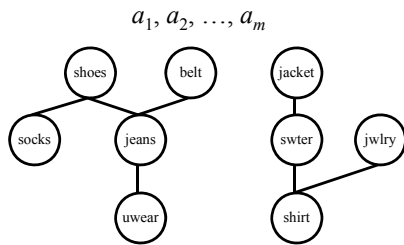
E.g. underwear is a new minimal element

✓ Now every element of this new poset cannot be a proper lower bound of a_1 and a_2 in the original poset

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This process continues until all elements are removed

We obtain a new order of the elements satisfying the given constraints:



The arrangement of the given poset in the new total order a_1, a_2, \dots compatible with the old order is called the Topological sorting

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