On the twisted Alexander polynomial and the A-polynomial of 2-bridge knots

Vu Q. Huynh and Thang T. Q. Le

Abstract. We show that the A-polynomial $A(L, M)$ of a 2-bridge knot $b(p, q)$ is irreducible if $p$ is prime, and if $(p - 1)/2$ is also prime and $q 
eq 1$ then the $L$-degree of $A(L, M)$ is $(p - 1)/2$. This shows that the AJ conjecture relating the A-polynomial and the colored Jones polynomial holds true for these knots, according to work of the second author. We also study relationships between the A-polynomial of a 2-bridge knot and a twisted Alexander polynomial associated with the adjoint representation of the fundamental group of the knot complement. We show that for twist knots the A-polynomial is a factor of the twisted Alexander polynomial.

1. Background and conventions

1.1. Representation variety. Let $K$ be a knot in $S^3$ and $X = S^3 \setminus K$ be its complement. Let $\pi = \pi_1(X)$ be the fundamental group of the complement. Let $R(\pi) = \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ be the set of representations of $\pi$ to $\text{SL}(2, \mathbb{C})$. This is a complex affine algebraic set, which is called the representation variety, although it might be a union of a finite number of (irreducible) algebraic varieties in the sense of algebraic geometry. The group $\text{SL}(2, \mathbb{C})$ acts on $R(\pi)$ by conjugation. The algebro-geometric quotient of $R(\pi)$ under this action is called the character variety of $\pi$, denoted by $X(\pi)$. The character of a representation $\rho$ is the map $\chi_\rho : \pi \to \mathbb{C}$ determined by $\chi_\rho(\gamma) = \text{tr} \rho(\gamma)$, for $\gamma \in \pi$. There is a bijection between $X(\pi)$ and the set of characters of representations of $\pi$.

1.2. The A-polynomial. Let $B = (\mu, \lambda)$ be a pair of meridian-longitude of the boundary torus of $X$. Let $R_U$ be the subset of $R(\pi)$ containing all representations $\rho$ such that $\rho(\mu)$ and $\rho(\lambda)$ are upper triangular matrices:

$$\rho(\mu) = \begin{pmatrix} M & * \\ 0 & M^{-1} \end{pmatrix}, \quad \rho(\lambda) = \begin{pmatrix} L & * \\ 0 & L^{-1} \end{pmatrix}$$

(any representation can be conjugated to have this form). Then $R_U$ is an algebraic set, because we only add the requirement that the lower left entries of $\rho(\mu)$ and $\rho(\lambda)$
are zeros. Define the projection map \( \xi : R_U \to \mathbb{C}^2 \) by \( \xi(\rho) = (L, M) \). Consider the Zariski closure \( \xi(R_U) \) of the projection \( \xi(R_U) \subset \mathbb{C}^2 \). It is known that \( \xi(R_U) \) is an algebraic set whose components have dimensions zero or one. If a component has dimension one then it is a curve defined by a single polynomial in \( L \) and \( M \). The product of these polynomials, divided by \( L \), is called the A-polynomial of \( K \). The reason for dividing by \( L \) is that if \( \rho(\lambda) \) is the identity matrix, therefore the component of \( \xi(R_U) \) corresponding to abelian representations is defined by a single equation \( L = 1 \). Thus in the construction of the A-polynomial one can restrict to nonabelian representations.

It is known that a multiple constant can be chosen so that the A-polynomial is an integer polynomial. We assume that the A-polynomial has no repeated factors; and that it has no integer factors, i.e. its coefficients are coprime. If instead of the basis \( B = (\mu, \lambda) \) we choose the other basis \( (\mu^{-1}, \lambda^{-1}) \) then the pair \( (L, M) \) is replaced by the pair \( (L^{-1}, M^{-1}) \) as can be seen from \( [1.1] \), and it is known that \( A_K(L^{-1}, M^{-1}) = \pm L^m M^n A_K(L, M) \). Thus \( A_K(L, M) \) is an integer polynomial defined up to a factor \( \pm L^m M^n \).

With finitely many exceptions, corresponding to a pair \( (L, M) \) satisfying \( A(L, M) = 0 \) there is a nonabelian representation \( \rho \in R(\pi) \) for which \( [1.1] \) holds.

For more on the A-polynomial we refer to [CCG+94], [CL96] and [CL98].

1.3. 2-bridge knots. Let \( p = 2n + 1 \), \( n \geq 1 \), and \( 0 < q < p \), \( q \) is odd, \( \gcd(p, q) = 1 \). The fundamental group of the complement \( X \) of the 2-bridge knot \( b(p, q) \) has a presentation \( \pi = \pi_1(X) = \langle a, b | wa = bw \rangle \), where both \( a \) and \( b \) are meridians. The word \( w \) has the form \( a^{e_1} b^{e_2} a^{e_1} b^{e_2} \ldots a^{e_{2n-1}} b^{e_{2n}} \), where \( e_i = (-1)^{\lfloor iq/p \rfloor} \). In particular, if we read \( w \) from right to left and interchange \( a \) and \( b \) then we get \( w \) again. For example, \( b(2n + 1, 1) \) is the torus knot \( T(2, 2n + 1) \), and in this case \( w = (ab)^n \).

We adopt the convention that if \( \rho \in R(\pi) \) and \( x \) is a word then we write \( \text{tr} x \) for \( \text{tr} \rho(x) \). Let \( x = \text{tr} a \) and \( y = \text{tr} ab \). Thang Le [Le93] showed that the character variety \( \chi^{\text{aff}}(\pi) \) of nonabelian representations of \( \pi \) is determined by the polynomial \( \Phi_{(p,q)}(x, y) = \text{tr} w - \text{tr} w' + \cdots + (-1)^{n-1} \text{tr} w^{(n-1)} + (-1)^n \), where if \( x \) is a word then \( x' \) denotes the word obtained from \( x \) by deleting the two letters at the two ends.

For more on 2-bridge knots see [BZ03], and for representations of 2-bridge knot groups we refer to [Ril84] and [Le93].

1.4. Nonabelian and irreducible representations. A representation \( \rho \) is said to be reducible if the action (i.e. the linear map) it induces on \( \mathbb{C}^2 \) fix a one dimensional
The trefoil as the 2-bridge knot \( b(3, 1) \).

This is equivalent to saying that \( \rho \) can be conjugated to be a representation by upper triangular matrices (one can take an eigenvector of the linear map as a new basis vector for \( \mathbb{C}^2 \)). Otherwise \( \rho \) is said to be irreducible.

An elementary argument (as suggested above) would show that if \( \rho \) is irreducible then it is nonabelian. For 2-bridge knots we have a stronger result (Le93): Except finitely many cases, a nonabelian representation is irreducible. The Zariski closure \( \overline{X^{irr}(\pi)} \) of the set of characters of irreducible representations is exactly the character variety \( X^{nab}(\pi) \) of nonabelian representations. Therefore in some arguments we can consider irreducible representations instead of nonabelian representations.

1.5. The A-polynomial of 2-bridge knots. Suppose that \( \rho \) is an irreducible representation. After conjugations if necessary we may assume that

\[
\rho(a) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} M & 0 \\ -z & M^{-1} \end{pmatrix}.
\]

We have \( x = \text{tr} a = M + M^{-1} \) and \( z = x^2 - 2 - y \) where \( y = \text{tr} ab \). Let \( \lambda = \tilde{w}w^{-2e} \), where \( \tilde{w} \) is the word obtained from \( w \) by writing the letters in \( w \) in reversed order (i.e. by interchanging \( a \) and \( b \)), and \( e \) is the sum of the exponents of the letters in \( w \). Then \( \lambda \) represents the longitude of the boundary torus of the knot complement, and we define \( \mathcal{L}(M, y) \) to be the upper left entry of the matrix \( \rho(\lambda) \). Then up to a factor of the form an integral power of \( M \), \( \mathcal{L}(M, y) \) is a polynomial. Because \( x = M + M^{-1} \) we can consider \( \Phi \) as a function in \( M \) and \( y \), up to a factor of the form \( M \) to an integral power it is a polynomial. The A-polynomial \( A(L, M) \) can be computed by deleting repeated factors from the resultant \( \text{Res}(\Phi(M, y), \mathcal{L}(M, y) - L) \), where the resultant is computed with respect to \( y \).

The description above can be implemented for computer calculations.
Example 1.1. The A-polynomial of $b(3, 1)$ (the trefoil) is $LM^0 + 1$, and that of $b(5, 3)$ (the figure-8 knot) is $-LM^8 + LM^6 + L^2M^4 + 2LM^4 + M^4 + LM^2 - L$.

For further details on the A-polynomial of 2-bridge knots we refer to [CCG+94] and [HS04].

1.6. The adjoint representation. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of $SL(2, \mathbb{C})$ consists of $2 \times 2$ matrices with zero traces. Consider the adjoint representation of $SL(2, \mathbb{C})$, $Ad : SL(2, \mathbb{C}) \rightarrow Aut(\mathfrak{sl}_2(\mathbb{C}))$. For $A \in SL(2, \mathbb{C})$ and $x \in \mathfrak{sl}_2(\mathbb{C})$ we have $Ad_A(x) = AxA^{-1}$. Since $\mathfrak{sl}_2(\mathbb{C})$ can be identified with $\mathbb{C}^3$, $Ad_A$ is a linear map on $\mathbb{C}^3$ and it turns out that it belongs to $SO(3, \mathbb{C})$. If $\rho \in R(\pi)$ then the composition $Ad \circ \rho$ is a representation of $\pi$ to $SO(3, \mathbb{C})$.

2. Irreducibility of the A-polynomial of 2-bridge knots

2.1. Introduction. In his recent study on the AJ conjecture which relates the A-polynomial and the colored Jones polynomial of a knot, Thang Le [Le04] proved that for a 2-bridge knot $b(p, q)$ the AJ conjecture holds true if the A-polynomial is irreducible and has $L$-degree $(p - 1)/2$. In this chapter we will provide a proof for the result (Theorem 2.5 below) that the above condition is satisfied if both $p$ and $(p - 1)/2$ are prime and $q \neq 1$.

In a related result, recently Hoste and Shanahan [HS04] using trace field theory have proved that the A-polynomial of the twist knot $K_n$, which is the 2-bridge knot $b(4n + 1, 2n + 1)$, is irreducible. From their recursive formula it can be checked easily that the $L$-degree is exactly 2n.

2.2. Proofs. Let $\Phi_n(x, y) = \Phi_{(p, 1)}(x, y)$, where $p = 2n + 1$. It has been shown in [Le93] Proposition 4.3.1] (also see below) that $\Phi_n(x, y)$ does not depend on $x$.

Proposition 2.1. $\Phi_n(y)$ is irreducible if and only if $2n + 1$ is prime.

Proof. It is immediate from [Le93] Proposition 4.3.1] that $\Phi_n(2y) = (T_n(y) + T_{n+1}(y))/(y + 1)$, where $T_n$ is the n-th Chebyshev polynomial (of the first kind). Let $\Phi_n(y) = \Phi_{n}(2y)$. It is well-known that by letting $\theta = \cos y$, we can write $T_n(y) = \cos(n\theta)$, and so $\Phi_n(\theta) = \cos ((2n+1)\theta)/\cos(\theta/2)$. It also follows that $\Phi_n(y)$ is an integer polynomial of degree $n$ with exactly $n$ roots given by $y = \cos (\frac{k+1}{2n+1}\pi), 0 \leq k \leq n - 1$. Fix $\theta = \pi/p$. Noting that $\Phi_n$ has no integer factor since $\Phi_n(0) = \pm 1$ we see that $\Phi_n$ is irreducible, and so is $\Phi_n$, if and only if the extension field degree $[\mathbb{Q}(\cos \theta) : \mathbb{Q}]$ is exactly the degree of $\Phi_n$.

Noticing that $\cos \theta = (e^{i\theta} + e^{-i\theta})/2$, we want to study the extension field $\mathbb{Q}(e^{i\theta})$. It is well-known (see, e.g. [Lan93 p. 276]) that the irreducible polynomial of $e^{i\theta}$ is the
cyclo
tic polynomial 
\[ C_{2p}(y) = \prod_{1 \leq d \leq 2p, (d, 2p) = 1} (x - e^{2\pi i/p}). \]
This is an integer polynomial whose degree is \( \varphi(2p) = \varphi(p) \), here \( \varphi \) is the Euler totient function. Thus the degree of the extension field is \([\mathbb{Q}(e^{2\pi i}) : \mathbb{Q}] = \varphi(p)\). From the identity \((x - e^{2\pi i})(x - e^{-2\pi i}) = x^2 - 2(\cos \theta)x + 1\), we see that \([\mathbb{Q}(e^{2\pi i}) : \mathbb{Q}(\cos \theta)] = 2\), thus \([\mathbb{Q}(\cos \theta) : \mathbb{Q}] = \varphi(p)/2\). Therefore \( \Phi_n \) is irreducible if and only if \( \varphi(p) = p - 1 \), which happens if and only if \( p \) is prime. □

**Proposition 2.2.** We have \( \Phi_{(p,q)}(0,y) = \Phi_{(p,1)}(y) \). Hence if \( \Phi_{(p,1)}(y) \) is irreducible then \( \Phi_{(p,q)}(x,y) \) is also irreducible.

*Proof.* Recall from section 1.5 that we can write \( \rho(a) = (M^1, 0) \) and \( \rho(b) = (M^{-1}, 0) \), where \( M + M^{-1} = x \) and \( z = x^2 - 2y \). If \( x = \text{tr}a = \text{tr}b = M + M^{-1} = 0 \) then it is immediate that \( \rho(a^{-1}) = -\rho(a) \) and \( \rho(b^{-1}) = -\rho(b) \) (this can also be seen from the Cayley-Hamilton Theorem: the characteristic polynomial of \( \rho(a) \) is \( t^2 - (\text{tr}a)t + 1 \).

Recall that \( \Phi_{(p,q)}(x,y) = \text{tr}w - \text{tr}w' + \cdots + (-1)^{n-1} \text{tr}w^{(n-1)} + (-1)^n \). Because the word \( w \) is palindromic, so is each word \( w^{(i)}, 0 \leq i \leq n - 1 \), and hence in \( w^{(i)} \) we have \( a^{-1} \) and \( b^{-1} \) appear in pairs. That means \( \rho(w^{(i)}) \) does not change if we replace \( a^{-1} \) by \( a \) and \( b^{-1} \) by \( b \). Thus \( \rho(w^{(i)}) = \rho((ab)^{n-i}) \). Recalling that for a torus knot \( b(p, 1) \) we have \( w = (ab)^n \), the result follows. □

Because \( x = M + M^{-1} \) we can consider \( \Phi \) as a function in \( M \) and \( y \), and it is a polynomial up to a factor of the form \( M \) to an integral power, which is omitted.

**Proposition 2.3.** If \( \Phi(M,y) \) is irreducible then \( A(L, M) \) is irreducible.

*Proof.* Recall from Section 1.5 that the A-polynomial \( A(L, M) \) of a 2-bridge knot can be computed by deleting repeated factors from \( \text{Res}(\Phi(M,y), \mathcal{L}(M,y) - L) \), where \( \mathcal{L}(M,y) \) is a polynomial and the resultant is computed with respect to \( y \).

We have \( A(L, M) = 0 \) if and only if there is \( y \) such that \( \Phi(M,y) = 0 \) and \( \mathcal{L}(M,y) = L \). Writing \( Z(f) \) for the zero set of a polynomial \( f \), we see that for each \( (M, L) \in Z(A(L, M)) \) there is \( (M, y) \in Z(\Phi(M,y)) \) such that \( (M, \mathcal{L}(M,y)) = (M, L) \).

In what follows we use some simple notions in algebraic geometry, which can be found for example in [Har77]. Consider the map \( pr : \mathbb{C}^2 \to \mathbb{C}^2 \) given by \( pr(u,v) = (u, \mathcal{L}(u,v)) \). This map is continuous under the Zariski topology. It projects \( Z(\Phi(M,y)) \) onto \( Z(A(L, M)) \).

Note that \( f \) is an irreducible polynomial if and only if \( Z(f) \) is an irreducible algebraic set. Now suppose that the A-polynomial is reducible, hence \( Z(A(L, M)) \) is a union of two nonempty closed subsets \( B \) and \( C \). Then \( pr^{-1}(B) \cap Z(\Phi) \) and
pr^{-1}(C) \cap Z(\Phi) are two nonempty closed sets whose union is Z(\Phi). This implies that \Phi(M, y) is reducible, a contradiction. \qed

Proposition 2.4. If the L-degree of A(L, M) is 1 then q = 1, and so b(p, q) is the torus knot T(2, p).

The idea for the following proof was communicated to us by Nathan Dunfield. We also thank Xingru Zhang for a discussion on this topic.

Proof. We need the concept of Newton polygons of A-polynomials. The Newton polygon of A(L, M) is the convex hull of the set of points (i, j) on the real LM-plane such that the coefficient \(a_{ij}\) of the term \(a_{ij}L^iM^j\) of A(L, M) is nonzero. The slopes of the sides of the Newton polygon are boundary slopes of incompressible surfaces in the knot complement (CCG+94).

For example the following figure shows the Newton polygon of the torus knot \(b(3, 1) = T(2, 3)\) (the trefoil) whose A-polynomial is \(LM^6 + 1\), and that of \(b(5, 3)\) (the figure-8 knot) whose A-polynomial is \(-LM^8 + LM^6 + L^2M^4 + 2LM^4 + M^4 + LM^2 - L\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{newton_polygons}
\caption{Newton polygons of the A-polynomials of \(b(3, 1)\) and \(b(5, 3)\).}
\end{figure}

Suppose that the L-degree of A(L, M) is 1. This means that the Newton polygon either has \(\infty\) as a slope, or has only one edge. The Hatcher-Thurston classification of incompressible surfaces in 2-bridge knot complements [HT85, Proposition 2] shows that actually \(\infty\) cannot be a slope, in fact all boundary slopes are integers.

Thus the Newton polygon has only one edge. For a hyperbolic knot the Newton polygon has at least two distinct sides. Thus the knot is non-hyperbolic.

Since 2-bridge knots are alternating ([BZ03]) a theorem of Menasco [Men84] says that the knot can only be a torus knot. Since the bridge number of a torus knot T(p, q) is at least \(\min\{p, q\}\), the torus knot must be \(T(2, p) = b(p, 1)\).
Note that for a torus knot \( T(2, p) \) indeed \( A(L, M) = LM^{2p} + 1 \) ([HS04, Zha04]) having \( L \)-degree 1.

**Theorem 2.5.** If \( p \) is prime then the A-polynomial of \( b(p, q) \) is irreducible. Furthermore if \((p - 1)/2 \) is also prime and \( q \neq 1 \) then the \( L \)-degree of \( A(L, M) \) is \((p - 1)/2 \).

**Proof.** The first part follows from Propositions 2.1, 2.2 and 2.3. We prove the second part.

First we claim that the \( y \)-degree of \( \Phi_{(p,q)}(M,y) \) is \( n = (p - 1)/2 \). Indeed, look at \( \Phi_{(p,q)}(M,y) = \text{tr} w - \text{tr} w + \cdots + (-1)^{n-1} \text{tr} w^{(n-1)} + (-1)^n \). Because the letter \( b \) appears \( n \) times in the word \( w \), the entries of the matrix \( \rho(w) \) have \( z \)-degrees, hence \( y \)-degrees, at most \( n \). So the \( y \)-degree of \( \Phi_{(p,q)}(M,y) \) is at most \( n \). On the other hand Proposition 2.2 and the proof of Proposition 2.1 show that the \( y \)-degree is at least \( n \), so the claim follows.

From the determinant description of resultant ([Lan93, p. 200]) it is clear that \( \text{Res}(\Phi(M,y), \mathcal{L}(M,y) - L) \) has degree \( n \) in \( L \). Since \( A(L, M) \) is irreducible we have a positive integer \( k \) such that \( A^k(M, L) = \text{Res}(\Phi(M,y), \mathcal{L}(M,y) - L) \). Thus the \( L \)-degree \( \ell \) of \( A(L, M) \) must be a factor of \( n \). If \( n \) is prime then \( \ell \) can only be 1 or \( n \). If \( \ell = 1 \) then the knot is a torus knot and \( q = 1 \) according to Proposition 2.4. \( \square \)

3. **Twisted Alexander polynomial and the A-polynomial of 2-bridge knots**

**Definition 3.1.** Let \( \pi = \langle a, b/r = waw^{-1}b^{-1} = 1 \rangle \). Let \( \rho \) be the representation of the free group \( \langle a, b \rangle \) defined by the formula

\[
\rho(a) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} M & 0 \\ -z & M^{-1} \end{pmatrix}.
\]

Extend the map \( Ad \circ \rho \) linearly, and consider \( M \) and \( z \) as formal variables. The twisted Alexander polynomial \( \Delta^d_K(M, z) \) associated to \( \pi \) is defined by

\[
\Delta^d_K(M, z) = \gcd(\det(Ad \circ \rho(\partial r/\partial a)), \det(Ad \circ \rho(\partial r/\partial b))) \in \mathbb{C}[M^{\pm 1}, z^{\pm 1}].
\]

It is a polynomial in \( M \) and \( z \) up to a factor \( \pm M^{m}z^{n} \).

For each pair \( (L_0, M_0) \) such that \( A_K(L_0, M_0) = 0 \) there is a finite number of numbers \( z_i \in \mathbb{C} \) such that both polynomial equations \( \Phi(M_0, z_i) = 0 \) and \( \mathcal{L}(M_0, z_i) = L_0 \) are satisfied.

**Proposition 3.2.** Except for finitely many pairs \( (L_0, M_0) \), if \( A_K(L_0, M_0) = 0 \) then \( \Delta^d_K(M_0, z_i) = 0 \).
Theorem asserts that if \( \rho \) vector space to a subspace of the first cohomology group \( H^1(X) \) of \( \partial X \) by the 6-matrix (3.2)

\[
\rho(a) = \begin{pmatrix} M_0 & 1 \\ 0 & M_0^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} M_0 & 0 \\ -z_t & M_0^{-1} \end{pmatrix}.
\]

Following a standard argument, the knot complement \( X \) is simple homotopic to a 2-dimensional cell complex with one 0-cell, two 1-cells and one 2-cell. Letting \( \widetilde{X} \) be the universal cover, we can consider the cochain complex of complex vector spaces:

\[
0 \leftarrow \mathbb{C}^3 \otimes \mathbb{Z}[\pi_1, \text{Ad}_{\rho}] C^2(\widetilde{X}) \xleftarrow{\partial_2} \mathbb{C}^3 \otimes \mathbb{Z}[\pi_1, \text{Ad}_{\rho}] C^1(\widetilde{X}) \xleftarrow{\partial_1} \mathbb{C}^3 \otimes \mathbb{Z}[\pi_1, \text{Ad}_{\rho}] C^0(\widetilde{X}) \leftarrow 0.
\]

Here \( \partial_2 \) is represented by the \( 3 \times 6 \)-matrix (\( \text{Ad}_{\rho}(\partial r / \partial a) \) \( \text{Ad}_{\rho}(\partial r / \partial b) \)) and \( \partial_1 \) is represented by the \( 6 \times 3 \)-matrix (\( \text{Ad}_{\rho}(b) - 1 \)). A direct computation shows that \( \text{Ad} \circ \rho(b - 1) \) is nonsingular. Thus \( \text{rank}(\text{Im} \partial_1) = 3 \). The first cohomology group with local coefficients of \( X \) is \( H^1_{\text{Ad}_{\rho}}(X) = \text{ker} \partial_2 / \text{Im} \partial_1 \).

At this point we use a theorem of Weil [Wei64] (see [Por97, p. 69], [BZ00]). The theorem asserts that if \( \rho \) is an irreducible representation then the Zariski tangent \( T^2_{\chi_{\rho}}(X(\pi)) \) of the character variety \( X(\pi) \) at the point \( \chi_{\rho} \) is isomorphic as complex vector space to a subspace of the first cohomology group \( H^1_{\text{Ad}_{\rho}}(X) \). For the Zariski tangent space at a point \( P \) of an algebraic variety \( Y \) we always have \( \text{rank} T^2_{\rho}(Y) \geq \text{rank}(Y) \). In this case because the point \( \chi_{\rho} \) arises from a point on the curve defined by \( A(L, M) \), the dimension of the irreducible component of \( X(\pi) \) containing \( \chi_{\rho} \) is at least one (we can also evoke a theorem of Thurston to this effect, see e.g. [CS83 Proposition 3.2.1]). Thus \( \text{rank} T^2_{\chi_{\rho}}(X(\pi)) \geq 1 \), hence \( \text{rank} H^1_{\text{Ad}_{\rho}}(X) \geq 1 \).

Since \( \text{rank}(\ker \partial_2 / \text{Im} \partial_1) \geq 1 \) and \( \text{rank}(\text{Im} \partial_1) = 3 \) it follows that \( \text{rank}(\ker \partial_2) \geq 4 \), hence \( \text{rank}(\text{Im} \partial_2) \leq 2 \). This means that both \( 3 \times 3 \)-matrices \( \text{Ad} \circ \rho(\partial r / \partial a) \) and \( \text{Ad} \circ \rho(\partial r / \partial b) \) have ranks less than three and thus are singular. Hence \( \text{det}(\text{Ad} \circ \rho(\partial r / \partial a)) = \text{det}(\text{Ad} \circ \rho(\partial r / \partial b)) = 0 \). This means \( \Delta_K^{\text{Ad}}(M, z) \) vanishes when it is evaluated at \( (M_0, z_t) \).

In the special case of a twist knot \( K_n \), which is the 2-bridge knot \( b(4n + 1, 2n + 1) \), it is shown in [HS04, p. 203] (note that \( K_n = J(2, -2n) \) in their notation) that the correspondence \( z_t \mapsto L_0 \) is one-to-one. Specifically \( z \) can be expressed in terms of \( L \) and \( M \) as

\[
z = \frac{(1 - L)(1 - M^2)}{L + M^2}.
\]

Using this change of variable we can write the twisted Alexander polynomial \( \Delta_K^{\text{Ad}}(M, z) \) as a polynomial \( \Delta_K^{\text{Ad}}(L, M) \).
Theorem 3.3. If \( K \) is twist knot then the polynomial \( A_K(L, M) \) is a factor of the polynomial \( \Delta^\text{Ad}_K(L, M) \).

Proof. For a twist knot Proposition 3.2 says that the zero set \( Z(A) \) of the A-polynomial \( A(L, M) \) minus a set \( I \) consists of finitely many points is contained in the zero set \( Z(\Delta^\text{Ad}) \) of the twisted Alexander polynomial \( \Delta^\text{Ad}(L, M) \). The Zariski closure of \( Z(A) \setminus I \) is exactly \( Z(A) \). Thus we have \( Z(A) \subset Z(\Delta^\text{Ad}) \) and so \( A(L, M) \) is a factor of \( \Delta^\text{Ad}(L, M) \). \( \square \)

References


FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF SCIENCE, VIETNAM NATIONAL UNIVERSITY, 227 NGUYEN VAN CU, DIST. 5, HO CHI MINH CITY, VIETNAM
E-mail address: hqvu@hcmus.eduvn

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332-0160, USA
E-mail address: letu@math.gatech.edu
URL: http://www.math.gatech.edu/~letu