Exercises: Groups

1. Determine whether each of the following systems forms a group \((G, \cdot)\), and which are abelian groups. Give reasons for your answers.

   a) \(G = \mathbb{Q} \setminus \{-6\}, x \cdot y = 90xy + 540x + 540y + 3234\).

   b) \(G = \mathbb{R}^+, x \cdot y = \ln(e^x + e^y - 1), \) where \(\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}\).

   c) \(G = \mathbb{R}, x \cdot y = x\sqrt{y^2 + 1} + y\sqrt{x^2 + 1}\).

   d) \(G = \mathbb{R} \times \mathbb{R}, (x_1, y_1) \cdot (x_2, y_2) = (x_1 + x_2, e^{x_2}y_1 + y_2)\).

   e) \(G = \mathbb{R}^* \times \mathbb{R}, (x_1, y_1) \cdot (x_2, y_2) = (x_1x_2, x_1y_2 + y_1), \) where \(\mathbb{R}^* = \mathbb{R} - \{0\}\).

   f) \(G = \mathbb{R}^+ \setminus \{1\}, x \cdot y = x^{\text{ln}y}\).

   g) \(G = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{Q}, a \neq 0\right\}, \) operation matrix multiplication.

2. Suppose \(G\) is a group such that \((ab)^2 = a^2b^2\) for every \(a, b \in G\). Show that \(G\) is abelian.

3. a) Show that \(a\) and \(a^{-1}\) are of the same order.

   b) Show that \(ab\) and \(ba\) are of the same order.

4. Let \(g\) be an element of order \(n\) in a group, and let \(m \geq 1\). Show that

   a) If \(\gcd(n, m) = d\), then \(g^m\) has order \(n/d\).

   b) In particular, if \(m\) divides \(n\), then \(gm\) has order \(n/m\).

5. Write the permutations as a product of disjoint cycles. Find the order of each permutation and state whether the permutation is even or odd.

   a) \(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 3 & 4 & 5 \end{pmatrix}\)

   b) \(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 1 & 5 & 7 & 3 \end{pmatrix}\)

   c) \(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 4 & 2 & 7 & 3 & 5 & 1 & 6 \end{pmatrix}\).

6. If a group \(G\) has only one element \(a\) of order 2, show that for every \(x\) in \(G\), \(xa = ax\).

7. Suppose \(|G| = p\) where \(p\) is prime. Show that \(G\) is cyclic.

8. If \(G\) is a group in which \(a^2 = e\), the identity for all \(a \in G\), show that \(G\) is abelian.
9. If $H$ and $K$ are subgroups of a group $G$, prove that $H \cap K$ is also a subgroup of $G$. Is $H \cup K$ necessarily a subgroup of $G$?

10. a) Show that $H = \left\{ \begin{pmatrix} x & y \\ 2y & x \end{pmatrix} | x, y \in \mathbb{Q} \right\}$ is a subgroup of $(\text{M}(2, \mathbb{Q}), +)$.
    b) Show that $H = \left\{ \begin{pmatrix} x & y \\ 2y & x \end{pmatrix} | x, y \in \mathbb{Q}; x^2 + y^2 > 0 \right\}$ is a subgroup of $(\text{GL}(2, \mathbb{Q}), \cdot)$.

11. Given $m, n \in \mathbb{Z}$. Prove that $m\mathbb{Z} \cap n\mathbb{Z} = \text{lcm}(m, n)\mathbb{Z}$ and $m\mathbb{Z} + n\mathbb{Z} = \text{gcd}(m, n)\mathbb{Z}$

12. a) Find all the group morphisms from $(\mathbb{Q}, +)$ to $(\mathbb{Z}, +)$.
    b) Find all the group morphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$.

13. a) Is $(\mathbb{Z}, +)$ isomorphic to $(\mathbb{Q}^*, \cdot)$, where $\mathbb{Q}^* = \mathbb{Q} - \{0\}$? Give reasons.
    b) Is $(\mathbb{R}, +)$ isomorphic to $(\mathbb{R}^+, \cdot)$, where $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$? Give reasons.

14. The center of a group $G$ is the set $Z(G) = \{ x \in G | xg = gx \text{ for all } g \in G \}$. Show that $Z(G)$ is an abelian subgroup of $G$.

15. In any group $(G, \cdot)$ the element $a^{-1}b^{-1}ab$ is called the commutator of $a$ and $b$. Let $G'$ be the subset of $G$ consisting of all finite products of commutators. Show that $G'$ is a normal subgroup of $G$. This is called the commutator subgroup. Also prove that $G/G'$ is abelian.

16. Let $H$ be subgroup of $G$ with only two right coset. Show that $H$ is a normal subgroup of $G$.

17. Suppose $H$ and $N$ are subgroups of $G$ with $N$ normal. Prove that $H \cap N$ is normal in $H$ and $H/ (H \cap N)$ is isomorphic to $HN/N$.

18. Let $\mathbb{C}^*$ be the group of nonzero complex numbers under multiplication and let $W$ be the multiplicative group of complex numbers of unit modulus. Describe $\mathbb{C}^*/W$.

19. Show that $\text{GL}(n, \mathbb{R}) / \text{SL}(n, \mathbb{R})$ is isomorphic to $\mathbb{R}^*$.

20. Show that every group of order $\leq 5$ is abelian.

21. Show that cyclic groups of the same order are isomorphic.

22. Prove that if $G$ and $H$ are finite groups whose orders are relatively prime, there is only one morphism from $G$ to $H$, the trivial one.

23. An automorphism of group $G$ is isomorphism $G \to G$. For a fixed $g \in G$, the mapping $\gamma_g : G \to G$, defined by $\gamma_g(x) = gxg^{-1}$, is an automorphism.
   i) Prove that $\text{Aut}(G)$, the set of all the automorphism of a group $G$, is group under composition.
   ii) Prove that $\gamma : G \to \text{Aut}(G)$, defined by $g \mapsto \gamma_g$, is a homomorphism.
   iii) Prove that $\ker \gamma = Z(G)$.
   iv) Prove that $\text{im} \gamma = \text{Aut}(G)$.